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New Fixed Points Theorems.

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Contents

Introduction

The celebrated Banach contraction principle (1922) plays an important role in various fields of applied mathematical analysis, statistics, chemistry, biology, computer science, engineering and economics in dealing with problems arising in approximation theory, potential theory, game theory, mathematical economics, theory of differential equations, theory of integral equations, theory of matrix. The classical Banach contraction principle is remarkable in its simplicity and it is perhaps the most widely applied fixed point theorem in all of analysis. This is because the contractive condition on the operator is easy to test and it requires only the structure of a complete metric space for its setting. It is known that Banach contraction principle has been used to solve the existence of solutions for nonlinear integral equations and nonlinear differential equations in Banach spaces. Also, it has been applied to study the convergence of algorithms in computational mathematics. Since then a number of generalizations in various directions of the Banach contraction principle have been investigated by several authors. It is known that common fixed point theorems are generalizations of fixed point theorems and hence, over the past few decades, many researchers were interested in generalizing fixed point theorems to coincidence point theorems and common fixed point theorems. Many mathematicians, as Caristi [23], gave a significant contribute working on the contractive condiction. Others, as Nadler [66], Mizoguchi and Takahashi [65], extended fixed point theory to multivalued mappings and improved the Banach contraction principle. In fact, it is possible to introduce new type of self-mappings (see for instance Kannan, Chatterjea [36]) or study fixed point theorems by changing the structure of the space itself. In particular, Branciari [20] introduced a concept of generalized metric space by replacing the triangle inequality with a more general inequality; clearly, any metric space is a generalized metric space but the converse is not true. For more, the reader can refer to [4, 16, 24, 31, 32, 39, 49, 52, 56, 64, 83, 84, 88]. On the other hand, in 1994, Matthews [63] introduced the notion of partial metric space as a part of the study of denotational semantics of dataflow networks, showing that the contraction principle can be generalized to the partial metric context for applications in program verification. Moreover, the existence of several connections between partial metrics and topological aspects of domain theory have been lately pointed by other authors as O'Neill [71], Bukatin and Scott [21], Bukatin and Shorina [22], Romaguera and Schellekens [82] and others (see also [3, 12, 25, 26, 43, 46, 47, 48, 51, 54, 55, 81, 89, 90] and the references therein). After the result of Matthews [63], the interest for fixed point theory developments in partial metric spaces has been constantly growing. Indeed, many authors presented significant contributions in the directions of establishing partial metric versions of well-known fixed point theorems in classical metric spaces (see for example [19, 27, 29]).

In this thesis we give new results that extend and generalize the previous results in the literature. We support this study by presenting application and examples. This thesis is devided in three chapters, each chapter containing several sections.

Chapter 1: This chapter is devoted to the study of fixed point theory in the setting of metric spaces. After introduction and preliminaries, we present the new theory. Also we study an initial-value problem for parabolic equations via fixed point methods.

Chapter 2: In this chapter we recall basic notions and properties of generalized metric spaces. Then, we present coincidence and common fixed point theorems in the setting of generalized metric spaces, with and without a partial order.

Chapter 3: In this chapter we consider partial metric spaces. After introduction and preliminaries, we give some fixed point results for single-valued and multi-valued mappings.

Research Papers:

We study a general initial-value problem for parabolic equations in Banach spaces, by using a monotone operator method and provide sufficient conditions for the existence of solution to such a type of problem. We give new fixed point theorems on metric spaces in the paper:

- *Vincenzo La Rosa, Calogero Vetro*

  Solution of an initial-value problem for parabolic equations via monotone operator methods.

Electronic Journal of Differential Equations, 2014, Vol. 2014, Article No. 225, 1-10.

We establish some common fixed point theorems for mappings satisfying an $\alpha - \psi - \varphi-$contractive condition in generalized metric spaces. Presented theorems extend and generalize many existing results in the literature. New fixed point theorems on Generalized Metric Spaces are published in:

- *Vincenzo La Rosa, Pasquale Vetro*

  Common fixed points for $\alpha-\psi-\varphi$***-contractions in generalized metric spaces***.

  Nonlinear Analysis: Modelling and Control, 2014, Vol. 19, No. 1, 43-54.

We establish some fixed point theorems for mappings satisfying Geraghty-type contractive conditions in the setting of partial metric spaces and ordered partial metric spaces. Presented theorems extend and generalize many existing results in the literature. Examples are given showing that these results are proper extensions of the existing ones. New Fixed point theorems on Partial Metric Spaces are published in:

- *Vincenzo La Rosa, Pasquale Vetro*

  Fixed points for Geraghty-contractions in partial metric spaces.

  Journal of Nonlinear Science and Applications, 2014, Vol. 7, No. 1, 1-10.

We establish results of fixed point for α_*-admissible mixed multi-valued mappings with respect to a function η and common fixed point for a pair (S, T) of mixed multi-valued mappings that is α_*-admissible with respect to a function η in partial metric spaces. An example is given to illustrate our result. New Fixed point theorems for multivalued mappings on Partial Metric Spaces are proposed in the article:

- *Vincenzo La Rosa, Pasquale Vetro*

  On fixed points for $\alpha - \eta - \psi$***-contractive multi-valued mappings in partial metric spaces****.

At the time the article is accepted by: *Nonlinear Analysis: Modelling and Control.*

New Fixed point theorems for partial-special multivalued mappings on Partial Metric Spaces are proposed in the article:

- *Vincenzo La Rosa*

  Geraghty's fixed point theorem for partial-special multi-valued mappings.

At the time the article is accepted by: *Thai Journal of Mathematics.*

Chapter 1

Fixed Points on Metric Spaces

Metric spaces are classical and natural settings for developing fixed point theory based on Banach contraction principle. After necessary introduction and preliminaries, we recall some well-known fixed point results in this context. Then, we give our results in metric spaces by using a transitive relation. We apply this theory to solve an initial-value problem for parabolic equations.

1.1 Metric Spaces

A metric space [93] is an ordered pair (X, d) where X is a set and d is a metric on X, i.e., a function $d : X \times X \to \mathbb{R}$ such that for any $x, y, z \in X$, the following conditions hold:

1.	$d(x, y) \geq 0$	(non-negative),
2.	$d(x, y) = 0 \Leftrightarrow x = y$,	(identity of indiscernibles),
3.	$d(x, y) = d(y, x)$,	(symmetry)
4.	$d(x, z) \leq d(x, y) + d(y, z)$	(triangle inequality).

The first condition follows from the other three, since: for any $x, y \in X$,
$d(x, y) + d(y, x) \geq d(x, x)$ (by triangle inequality) $\Rightarrow$
$d(x, y) + d(x, y) \geq d(x, x)$ (by symmetry) $\Rightarrow$
$2d(x, y) \geq 0$ (by identity of indiscernibles) $\Rightarrow d(x, y) \geq 0$.

The function d is also called distance function or simply distance. Often, d is omitted and one just writes X for a metric space if it is clear from the context what metric is used.

Let X be a metric space with metric d. If x_0 is a point of X and r is a positive real number, the open ball $S_r(x_0)$ with center x_0 and radius r is the subset of X defined by:

$$S_r(x_0) = \{x \in X : d(x_0, x) < r\}.$$

The closed ball $S_r[x_0]$ is defined by:

$$S_r[x_0] = \{x \in X : d(x_0, x) \leq r\},$$

where r is a nonnegative real number. A subset G of the metric space X is called an *open set*, if given any point x in G, there exists a positive number r such that $S_r(x_0) \subset G$. A subset F of X is called *closed set*, if the complement F^c of F is open.

Let X and Y be two metric spaces with metrics d_1 and d_2, and let f be a mapping of X into Y. f is said to be *continuous at the point* x_0 in X if for each $\epsilon > 0$, there exists $\delta > 0$ such that:

$$x \in X,\ d_1(x, x_0) < \delta \Rightarrow d_2(fx, fx_0) < \epsilon.$$

A mapping of X into Y is said to be *continuous* if it is continuous at each point in its domain X.

A mapping f of X into Y is called *nonexpansive* if

$$d_2(fx, fy) \leq d_1(x, y) \quad \forall \quad x, y \in X.$$

A mapping f of X into Y is called *contraction* if there exists a nonnegative number $r < 1$ with the property that

$$d_2(fx, fy) \leq r d_1(x, y) \quad \forall \quad x, y \in X.$$

It is obvius that such mappings are continuous. Let X be a metric space with metric d, and let $\{x_n\}$ be a sequence of points in X. We say that $\{x_n\}$ is *convergent* if there exists a point x in X such that for each $\epsilon > 0$, we can find a positive integer n_0 such that

$$d(x_n, x) < \epsilon \quad \forall \quad n \geq n_0.$$

We usually symbolize this by writing $x_n \to x$, or $\lim_{n \to +\infty} x_n = x$. The point x is called the *limit* of the sequence $\{x_n\}$. A sequence $\{x_n\}$ in X is a *Cauchy sequence* if for each $\epsilon > 0$, there exists a positive integer n_0 such that

$$d(x_n, x_m) < \epsilon \quad \forall \quad n, m \geq n_0.$$

It is obvious that every convergent sequence is a Cauchy sequence. A *complete metric space* is a metric space in which every Cauchy sequence is convergent.

For any system of roads and terrains the distance between two locations can be defined as the length of the shortest route connecting those locations. To be a metric there shouldn't be any one-way roads. The triangle inequality expresses the fact that detours aren't shortcuts. The examples below can be seen as concrete versions of this general idea.

- The real numbers with the distance function $d(x, y) = |y - x|$ given by the absolute difference, and more generally Euclidean n-space with the Euclidean distance, are complete metric spaces. The rational numbers with the same distance also form a metric space, but are not complete.
- The positive real numbers with distance function $d(x, y) = |\log(y/x)|$ is a complete metric space.
- Any normed vector space is a metric space by defining $d(x, y) = \|y - x\|$, see also metrics on vector spaces.

Definition 1.1 *We call **Banach space** a complete normed vector space.*

Now we give the elementary results [93] for the *lower semicontinuous functions*; these results are essential for studying nonlinear functional analysis, particulary, convex analysis.

Definition 1.2 *Let X be a topological space and let f be a function of X into $]-\infty, +\infty]$. Then, f is said to be lower semicontinuous on X if for any real number a, the set $\{x \in X : fx \leq a\}$ is closed in X.*

Theorem 1.1 ([93] Theorem 1.3.1) *Let X be a compact space and let $f : X \to]-\infty, +\infty]$ be a lower semicontinuous function. Then, there exists an element $x_0 \in X$ such that:*

$$fx_0 = \min\{fx : x \in X\}.$$

A function f of X into $]-\infty, +\infty]$ is said bounded below if there exists a real number M such that $M \leq fx$ for all $x \in X$.

Theorem 1.2 ([93] Theorem 1.3.4) *Let X be a topological space, let f, g be lower semicontinuous functions of X into $]-\infty, +\infty]$ and let α be a nonnegative number. Then the functions $f + g$ and αf defined by*

$$(f + g)x = fx + gx, \quad (\alpha f)x = \alpha fx \quad \forall \quad x \in X$$

are lower semicontinuous.

1.2 Classical Fixed Point Theorems

Theorem 1.3 ([17] Banach contraction principle) *Let (X, d) be a complete metric space and $f : X \to X$ be a contraction. Then f has a unique fixed point $z \in X$.*

The contraction definition implies that f is uniformly continuous which is a very strong condition. It is quite natural to ask whether the inequality can be replaced with another inequality which do not force f to be continuous. This question was answered affirmatively by Kannan:

Theorem 1.4 ([36] Kannan) *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping. Suppose that there exists $\gamma \in [0, \frac{1}{2}[$ such that*

$$d(fx, fy) \leq \gamma(d(x, fx) + d(y, fy))$$

for all $x, y \in X$. Then, f has a unique fixed point in X.

The same holds for the class of Chatterjea mappings. In fact, we have the following theorem:

Theorem 1.5 ([36] Chatterjea) *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping. Suppose that there exists $\gamma \in [0, \frac{1}{2}[$ such that*

$$d(fx, fy) \leq \gamma(d(x, fy) + d(y, fx))$$

for all $x, y \in X$. Then, f has a unique fixed point in X.

Definition 1.3 *Let S denotes the class of functions $\beta : [0, +\infty[\to [0, 1[$ which satisfy the condition $\beta(t_n) \to 1 \Rightarrow t_n \to 0$.*

The following generalization of Banach's contraction principle, proved in 1973, is due to Geraghty.

Theorem 1.6 ([40] Geraghty) *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping. Assume that there exists $\beta \in S$ such that, for all $x, y \in X$,*

$$d(fx, fy) \leq \beta(d(x, y))d(x, y).$$

Then f has a unique fixed point $z \in X$ and, for any choice of the initial point $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_n = fx_{n-1}$ for each $n \geq 1$ converges to the point z.

Very recently, Amini-Harandi and Emami proved the following existence theorem:

Theorem 1.7 ([8] Theorem 2.1) *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be an increasing mapping such that there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that there exists $\beta \in S$ such that*

$$d(fx, fy) \leq \beta(d(x, y))d(x, y)$$

for all $x, y \in X$ with $x \succeq y$. Assume that either f is continuous or X is such that if an increasing sequence $\{x_n\}$ converges to x, then $x_n \preceq x$ for each $n \geq 1$. Besides, if for all $x, y \in X$, there exists $z \in X$ which is comparable to x and y. Then f has a unique fixed point in X.

Let (X, d) be a metric space and let $CB(X)$ denote the collection of all nonempty closed and bounded subsets of X. For $A, B \in CB(X)$, define

$$H(A, B) := \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}, \tag{1.1}$$

where $d(x, A) := \inf\{d(x, a) : a \in A\}$ is the distance of a point x to the set A. It is known that H is a metric on $CB(X)$, called the Hausdorff metric induced by the metric d.

Definition 1.4 *Let (X, d) be a metric space. An element x in X is said to be a fixed point of a multi-valued mapping $T : X \to CB(X)$ if $x \in Tx$.*

We recall that $T : X \to CB(X)$ is said to be a multi-valued contraction mapping if there exists $k \in [0, 1[$ such that

$$H(Tx, Ty) \leq kd(x, y) \quad \forall \quad x, y \in X. \tag{1.2}$$

The study of fixed points for multi-valued contractions using the Hausdorff metric was initiated by Nadler [66] who proved the following theorem.

Theorem 1.8 ([66], Theorem 5) *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multi-valued contraction mapping. Then there exists $x \in X$ such that $x \in Tx$.*

1.3 New Fixed Point Theorems

Let X be a nonempty set. In the sequel $\mathcal{M}$ denotes a transitive relation on X, that is, $\mathcal{M}$ is a subset of $X \times X$ such that $(x, z) \in \mathcal{M}$ whenever $(x, y), (y, z) \in \mathcal{M}$. Let $f : X \to X$ be a mapping and $\mathcal{M}$ a subset of $X \times X$. The set $\mathcal{M}$ is f-invariant if $(fx, fy) \in \mathcal{M}$ whenever $(x, y) \in \mathcal{M}$.

Example 1.1 *Let $\preceq$ be a partial order on X such that $(X, \preceq)$ is a partially ordered set. Then*

$$\mathcal{M} = \{(x, y) \in X \times X : \ x \preceq y\}$$

is a transitive relation on X. Also if $f : X \to X$ is a nondecreasing mapping, then the set $\mathcal{M}$ is f-invariant.

In 1973, Geraghty gave Theorem 1.6 using the class of functions of Definition 1.3. Gordji et al. [42] generalized Geraghty's theorem using the function ψ (Definition 2.3). Then, he proved the following result:

Theorem 1.9 *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be a nondecreasing mapping such that there exists $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that there exist $\beta \in S$ and subadditive $\psi \in \Psi$ such that*

$$\psi(d(fx, fy)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))$$

for all $x, y \in X$ with $x \succeq y$. Assume that either f is continuous or X is such that if an increasing sequence $\{x_n\}$ converges to x, then $x_n \preceq x$ for each $n \in \mathbb{N}$. Then f has a fixed point. Besides, if for all $x, y \in X$, there exists $z \in X$ which is comparable to x and y, then f has a unique fixed point in X.

Now we present a generalization which will be extended in this chapter. Notice that, unlike Gordji et al. [42], we do not assume that the function ψ (Definition 2.3) is subadditive. For our further use, we state also the following lemma.

Lemma 1.1 *Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X such that:*

$$\lim_{n \to +\infty} d(x_{n+1}, x_n) = 0.$$

If $\{x_{2n}\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and two sequences $\{m_k\}$, $\{n_k\}$ of positive integers, with $m_k < n_k$, such that the following four sequences

$$\{d(x_{2m_k}, x_{2n_k})\}, \{d(x_{2m_k}, x_{2n_k+1})\}, \{d(x_{2m_k-1}, x_{2n_k})\}, \{d(x_{2m_k-1}, x_{2n_k+1})\}$$

tend to ϵ as $k \to +\infty$.

Notice that assertions similar to Lemma 1.1 (see, e.g. [74]) were used (and proved) in the course of proofs of some fixed point theorems in various papers. Besides, if Lemma 3.3 is true then, in particular, Lemma 1.1 is also true.

We give successively two existence results with and without continuity hypothesis.

Let $f : X \to X$ be a mapping and denote

$$M(x, y) = \max \left\{ d(x, y), d(x, fx), d(y, fy), \frac{1}{2}\left[d(x, fy) + d(fx, y)\right] \right\}$$

for all $x, y \in X$.

In the first theorem, we use the continuity hypothesis of f.

Theorem 1.10 *Let (X, d) be a complete metric space endowed with a transitive relation $\mathcal{M}$ on X and $f : X \to X$ be a mapping. Assume that the following conditions hold:*

(i) there exist $\beta \in S$ and $\psi \in \Psi$ such that

$$\psi(d(fx, fy)) \leq \beta(M(x, y))\psi(M(x, y)) \tag{1.3}$$

for all $(x, y) \in \mathcal{M}$ with $x \neq y$;

(ii) there exists $x_0 \in X$ such that $(x_0, fx_0) \in \mathcal{M}$;

(iii) $\mathcal{M}$ is f-invariant;

(iv) f is continuous.

Then f has a fixed point.

Proof. Let $x_0 \in X$ such that $(x_0, fx_0) \in \mathcal{M}$. We consider the sequence $\{x_n\}$ defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n-1} = x_n$ for some $n \in \mathbb{N}$, then x_{n-1} is a fixed point of f and the existence of a fixed point is proved. Now, we suppose that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. From $(x_0, x_1) = (x_0, fx_0) \in \mathcal{M}$, since $\mathcal{M}$ is f-invariant, we deduce $(x_1, x_2) = (fx_0, fx_1) \in \mathcal{M}$. This implies

$$(x_{n-1}, x_n) \in \mathcal{M} \quad \text{for all } n \in \mathbb{N}. \tag{1.4}$$

Now, using (1.3) with $x = x_{n-1}$ and $y = x_n$, we have

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &= \psi(d(fx_{n-1}, fx_n)) \\ &\leq \beta(M(x_{n-1}, x_n))\psi(M(x_{n-1}, x_n)), \end{aligned} \tag{1.5}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, fx_{n-1}), d(x_n, fx_n), \\ &\quad \frac{1}{2}[d(x_{n-1}, fx_n) + d(fx_{n-1}, x_n)]\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}d(x_{n-1}, x_{n+1})\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

If $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$, by (1.3), we have

$$\psi(d(x_n, x_{n+1})) \leq \beta(d(x_n, x_{n+1}))\, \psi(d(x_n, x_{n+1})) < \psi(d(x_n, x_{n+1}))$$

which is a contradiction. Then $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$. By (1.5), we get

$$\psi(d(x_n, x_{n+1})) \leq \beta(d(x_{n-1}, x_n))\psi(d(x_{n-1}, x_n)) < \psi(d(x_{n-1}, x_n)). \tag{1.6}$$

Thus $\{d(x_{n-1}, x_n)\}$ is a decreasing sequence of nonnegative numbers and hence there exists

$$\lim_{n \to +\infty} d(x_n, x_{n+1}) = r \geq 0.$$

Assume $r > 0$. Since $\psi(d(x_{n-1}, x_n)) \neq 0$ for all $n \in \mathbb{N}$, from (1.6) we deduce

$$\frac{\psi(d(x_n, x_{n+1}))}{\psi(d(x_{n-1}, x_n))} \leq \beta(d(x_{n-1}, x_n)) \leq 1 \quad \text{for all } n \in \mathbb{N}. \tag{1.7}$$

Letting $n \to +\infty$ in (1.7), by the continuity of the function ψ, we obtain

$$\lim_{n \to +\infty} \beta(d(x_{n-1}, x_n)) = 1.$$

On the other hand, since $\beta \in S$, we have $\lim_{n\to+\infty} d(x_{n-1}, x_n) = 0$ and so $r = 0$. Now, we show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. This implies that $\{x_{2n}\}$ is not a Cauchy sequence. Since $\mathcal{M}$ is a transitive relation, from $(x_{n-1}, x_n) \in \mathcal{M}$ for all $n \in \mathbb{N}$, we deduce that $(x_m, x_n) \in \mathcal{M}$ for all $m, n \in \mathbb{N}$ with $m < n$. If ϵ, $\{m_k\}$ and $\{n_k\}$ are as in Lemma 1.1, using (1.3) with $x = x_{2m_k-1}$ and $y = x_{2n_k}$ obviously we can assume that $x_{2m_k-1} \neq x_{2n_k}$, it follows that

$$\psi(d(x_{2m_k}, x_{2n_k+1})) \leq \beta(M(x_{2m_k-1}, x_{2n_k}))\psi(M(x_{2m_k-1}, x_{2n_k})) \qquad (1.8)$$

where

$$\begin{aligned} M(x_{2m_k-1}, x_{2n_k}) &= \max\{d(x_{2m_k-1}, x_{2n_k}), d(x_{2m_k-1}, x_{2m_k}), \\ &\quad d(x_{2n_k}, x_{2n_k+1}), \frac{1}{2}[d(x_{2m_k-1}, x_{2n_k+1}) + d(x_{2m_k}, x_{2n_k})]\}. \end{aligned}$$

Then, for $k \to +\infty$, we obtain $M(x_{2m_k-1}, x_{2n_k}) \to \epsilon$. We can assume $\psi(M(x_{2m_k-1}, x_{2n_k})) > 0$ for all $k \in \mathbb{N}$. From (1.8), we have

$$\frac{\psi(d(x_{2m_k}, x_{2n_k+1}))}{\psi(M(x_{2m_k-1}, x_{2n_k}))} \leq \beta(M(x_{2m_k-1}, x_{2n_k})) \leq 1$$

for all $k \in \mathbb{N}$. Now, letting $k \to +\infty$ in the previous inequality, by the continuity of the function ψ and $M(x_{2m_k-1}, x_{2n_k}) \to \epsilon$, we get

$$\lim_{k\to+\infty} \beta(M(x_{2m_k-1}, x_{2n_k})) = 1.$$

Since $\beta \in S$, we have

$$\lim_{k\to+\infty} M(x_{2m_k-1}, x_{2n_k}) = 0$$

a contradiction and hence $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, there exists $z \in X$ such that $\lim_{n\to+\infty} x_n = z$.

If f is a continuous mapping, then

$$z = \lim_{n\to+\infty} x_{n+1} = \lim_{n\to+\infty} fx_n = fz$$

and hence $fz = z$, that is, z is a fixed point of f. □

In the next theorem, we omit the continuity hypothesis of f.

Theorem 1.11 *Let (X, d) be a complete metric space endowed with a transitive relation $\mathcal{M}$ on X and $f : X \to X$ be a mapping. Assume that the following conditions hold:*

(i) there exist $\beta \in S$ and $\psi \in \Psi$ such that

$$\psi(d(fx, fy)) \leq \beta(M(x, y))\psi(M(x, y)) \tag{1.9}$$

for all $(x, y) \in \mathcal{M}$ with $x \neq y$;

(ii) there exists $x_0 \in X$ such that $(x_0, fx_0) \in \mathcal{M}$;

(iii) $\mathcal{M}$ is f-invariant;

(iv) if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in \mathcal{M}$ for all $n \in \mathbb{N}$ and $x_n \to z \in X$ as $n \to +\infty$, then $(x_n, z) \in \mathcal{M}$ for all $n \in \mathbb{N}$.

Then f has a fixed point.

Proof. Following the proof of Theorem 1.10, we know that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d). Then, there exists $z \in X$ such that $x_n \to z$ as $n \to +\infty$. On the other hand, from (1.4) and the hypothesis (iv), we have

$$(x_n, z) \in \mathcal{M} \quad \text{for all } n \in \mathbb{N}.$$

We assume that $z \neq fz$. From $x_n \neq x_{n+1}$ follows that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \neq z$ for all $n \in \mathbb{N}$. Using (1.9) with $x = x_{n_k}$ and $y = z$, we get

$$\psi(d(fx_{n_k}, fz)) \leq \beta(M(x_{n_k}, z))\psi(M(x_{n_k}, z)) < \psi(M(x_{n_k}, z)) \tag{1.10}$$

where

$$\begin{aligned} M(x_{n_k}, z) &= \max\Big\{d(x_{n_k}, z), d(x_{n_k}, fx_{n_k}), d(z, fz), \\ &\qquad \frac{1}{2}[d(x_{n_k}, fz) + d(z, fx_{n_k})]\Big\} \\ &= \max\Big\{d(x_{n_k}, z), d(x_{n_k}, x_{n_k+1}), d(z, fz), \\ &\qquad \frac{1}{2}[d(x_{n_k}, z) + d(z, fz) + d(z, x_{n_k+1})]\Big\}. \end{aligned}$$

Since $d(x_{n_k}, z), d(x_{n_k}, x_{n_k+1}) \to 0$ as $k \to +\infty$, for k great enough, we have

$$M(x_{n_k}, z) = d(z, fz).$$

Thus from (1.10), we obtain

$$\psi(d(fx_{n_k}, fz)) \leq \beta(d(z, fz))\psi(d(z, fz)) \tag{1.11}$$

for k great enough. Letting $k \to +\infty$ in (1.11), by the continuity of the function ψ, we have

$$\psi(d(z, fz)) \leq \beta(d(z, fz))\psi(d(z, fz)) < \psi(d(z, fz))$$

Therefore, we get $fz = z$ and this completes the proof. □

Thus, by using Theorems 1.10 and 1.11, we are able to establish the existence of a fixed point. Next step is to give sufficient conditions for obtaining uniqueness. Precisely, we will consider the following hypothesis:

(U): For all $(x, y) \notin \mathcal{M}$ there exists $z \in X$ such that $(x, z), (y, z) \in \mathcal{M}$ and

$$\lim_{n\to+\infty} d(f^{n-1}z, f^n z) = 0.$$

Theorem 1.12 *Adding condition (U) to the hypotheses of Theorem 1.10 (resp. Theorem 1.11) we obtain uniqueness of the fixed point of f.*

Proof. Suppose that x and y, with $x \neq y$, are two fixed points of f. If $(x, y) \in \mathcal{M}$, by using (1.3) we have

$$\begin{aligned} \psi(d(x, y)) &= \psi(d(fx, fy)) \leq \beta(M(x, y))\psi(M(x, y)) \\ &= \beta(d(x, y))\psi(d(x, y)) \\ &< \psi(d(x, y)), \end{aligned}$$

which is a contradiction and hence $x = y$. If $(x, y) \notin \mathcal{M}$, from (U), there exists $z \in X$ such that $(x, z), (y, z) \in \mathcal{M}$. Put $z_n = f^n z$ for all $n \in \mathbb{N}$. Since $\mathcal{M}$ is f-invariant, we have $(x, z_n), (y, z_n) \in \mathcal{M}$ for all $n \in \mathbb{N}$. Now, using (1.9) for all $n \in \mathbb{N}$ such that $z_n \neq \{x, y\}$, we obtain

$$\psi(d(x, z_{n+1})) = \psi(d(fx, fz_n)) \leq \beta(M(x, z_n))\psi(M(x, z_n)) \tag{1.12}$$

and

$$\psi(d(y, z_{n+1})) = \psi(d(fy, fz_n)) \leq \beta(M(y, z_n))\psi(M(y, z_n)). \tag{1.13}$$

Step 1. Assume that there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \to x$. If $\{z_{n_k}\}$ has a subsequence that converges to y, in this case we can assume that $z_{n_k} \to y$, then from $d(x,y) \leq d(x,z_{n_k}) + d(z_{n_k},y)$ letting $k \to +\infty$, we obtain $d(x,y) = 0$, that is $x = y$. Now, we assume that $d(y,z_{n_k}) > 0$ for all $k \in \mathbb{N}$. From (1.13), we have

$$\frac{\psi(d(y,z_{n_k+1}))}{\psi(M(y,z_{n_k}))} \leq \beta(M(y,z_{n_k})) \leq 1 \tag{1.14}$$

for all $k \in \mathbb{N}$, where

$$\begin{aligned} M(y,z_{n_k}) &= \max\{d(y,z_{n_k}), d(y,fy), d(z_{n_k},fz_{n_k}), \\ &\quad \frac{1}{2}[d(y,fz_{n_k}) + d(z_{n_k},fy)]\} \\ &= \max\{d(y,z_{n_k}), d(z_{n_k},z_{n_k+1}), \frac{1}{2}[d(y,z_{n_k+1}) + d(z_{n_k},y)]\}. \end{aligned}$$

Using the continuity of the function ψ, letting $k \to +\infty$ in (1.14), we get

$$\lim_{k\to+\infty} \beta(M(y,z_{n_k})) = 1$$

that implies

$$d(y,x) = \lim_{k\to+\infty} M(y,z_{n_k}) = 0$$

a contradiction and hence $x = y$. The same holds if there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \to y$.

Step 2. We consider the case that there exist $\epsilon > 0$ and $n(\epsilon) \in \mathbb{N}$ such that $d(x,z_n) \geq \epsilon$ for all $n \geq n(\epsilon)$. From condition (U),

$$\begin{aligned} M(x,z_n) &= \max\{d(x,z_n), d(x,fx), d(z_n,fz_n), \frac{1}{2}[d(x,fz_n) + d(z_n,fx)]\} \\ &= \max\{d(x,z_n), d(z_n,z_{n+1}), \frac{1}{2}[d(x,z_{n+1}) + d(z_n,x)]\} \end{aligned}$$

and (1.12), we deduce that $M(x,z_n) = d(x,z_n)$ for n great enough. Consequently, by (1.12), the sequence $\{d(x,z_n)\}$ for n great enough is decreasing and hence $d(x,z_n) \to r \geq 0$. Assume $r > 0$. Using the continuity of the function ψ, letting $n \to +\infty$ in (1.12), we get

$$\lim_{n\to+\infty} \beta(M(x,z_n)) = 1$$

that implies

$$r = \lim_{n\to+\infty} M(x,z_n) = 0$$

a contradiction and hence $r = 0$. Similarly, one can prove that $d(y, z_n) \to 0$ and hence $d(x, y) = 0$, that is, $x = y$.

$\square$

Proceeding as in the proof of Theorems 1.10 and 1.12, we obtain the following theorem; to avoid repetitions the details are omitted.

Theorem 1.13 *Let (X, d) be a complete metric space endowed with a transitive relation $\mathcal{M}$ on X and $f : X \to X$ be a mapping. Assume that the following conditions hold:*

(i) there exist $\beta \in S$ and $\psi \in \Psi$ such that

$$\psi(d(fx, fy)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))$$

for all $(x, y) \in \mathcal{M}$ with $x \neq y$;

(ii) there exists $x_0 \in X$ such that $(x_0, fx_0) \in \mathcal{M}$;

(iii) $\mathcal{M}$ is f-invariant;

(iv) f is continuous.

Then f has a fixed point. In addition, the fixed point is unique provided that

(v) for all $(x, y) \notin \mathcal{M}$ there exists $z \in X$ such that $(x, z), (y, z) \in \mathcal{M}$.

Analogously, proceeding as in the proof of Theorems 1.11 and 1.12, we obtain the following theorem.

Theorem 1.14 *Let (X, d) be a complete metric space endowed with a transitive relation $\mathcal{M}$ on X and $f : X \to X$ be a mapping. Assume that the following conditions hold:*

(i) there exist $\beta \in S$ and $\psi \in \Psi$ such that

$$\psi(d(fx, fy)) \leq \beta(\psi(d(x, y)))\psi(d(x, y)) \tag{1.15}$$

for all $(x, y) \in \mathcal{M}$ with $x \neq y$;

(ii) there exists $x_0 \in X$ such that $(x_0, fx_0) \in \mathcal{M}$;

(iii) $\mathcal{M}$ is f-invariant;

(iv) if $\{x_n\}$ *is a sequence in* X *such that* $(x_n, x_{n+1}) \in \mathcal{M}$ *for all* $n \in \mathbb{N}$ *and* $x_n \to z \in X$ *as* $n \to +\infty$*, then* $(x_n, z) \in \mathcal{M}$ *for all* $n \in \mathbb{N}$.

Then f *has a fixed point. In addition, the fixed point is unique provided that*

(v) for all $(x, y) \notin \mathcal{M}$ *there exists* $z \in X$ *such that* $(x, z), (y, z) \in \mathcal{M}$.

Notice that Theorems 1.13 and 1.14 are generalizations of Theorem 1.9 of Gordji et al. [42]. In fact, we get Theorem 1.9 if we choose the set $\mathcal{M}$ as in Example 1.1. Also, from Theorem 1.13, we deduce the result of Geraghty (Theorem 1.6) if we choose $\mathcal{M} = X \times X$ and $\psi(t) = t$.

1.4 Initial-Value Problem for Parabolic Equations

Over the years, the theory of differential equations is well investigated and consequently the methods developed for their solution are strongly related to the particular equation, see for instance [38]. In [38], Evans furnishes a comprehensive study on this topic. In this paper, referring to heat equation variants, we consider the existence of solutions for the following initial-value problem for parabolic equations:

$$\begin{cases} u_t(x,t) = q(x,t,u(x,t)) + k u_{xx}(x,t), & x \in \mathbb{R}, \quad t \in]0,T], \\ u(x,0) = \varphi(x), & x \in \mathbb{R}, \end{cases} \tag{1.16}$$

where we assume that $q : \mathbb{R} \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\varphi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, φ and φ' are bounded and $k, T > 0$.

Then, a function $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ is a solution of the parabolic equation in (1.16) if:

(s_1) $u \in C(\mathbb{R} \times [0,T])$;

(s_2) $u_t, u_{xx} \in C(\mathbb{R} \times [0,T])$;

(s_3) u is bounded in $\mathbb{R} \times [0,T]$;

(s_4) $u_t(x,t) = q(x,t,u(x,t)) + u_{xx}(x,t)$ for all $(x,t) \in \mathbb{R} \times [0,T]$.

The existence and uniqueness of solutions for general initial-value problems on some (possibly small) interval have been studied extensively, see for instance [13, 91] and the references therein. Practically, researchers are interested in establishing how large this interval might be and how solutions

of initial-value problems change when the differential equation or initial conditions are perturbed. In this direction, some results are obtained by using simple notions and techniques of fixed point theory. For instance, the well-known Banach contraction principle is an important tool for studying the existence and uniqueness of fixed points of certain mappings in metric spaces. Also, it provides a constructive method to find those fixed points. Finally, various applications to matrix equations, ordinary differential equations, and integral equations were presented by using this principle and its generalizations and extensions, see for instance [15, 17, 67, 75].
Thus, we give a generalization of a fixed point theorem for monotone mappings, due to Gordji et al. [42], in the setting of complete metric spaces endowed with a transitive relation. Then, by combining our result with Green's function formalism, we discuss the existence of solution for the initial-value problem (1.16) via monotone operator methods. In this section, we adapt the calculations in [42] to our situation.

Precisely, by using Theorem 1.14, we study the existence of solution for the initial-value problem (1.16). It is well-know that this problem, see [91], is equivalent to the integral equation:

$$\begin{aligned} u(x,t) &= \int_0^t d\tau \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4k(t-\tau)}}}{\sqrt{4\pi k(t-\tau)}}\, q(\xi,\tau,u(\xi,\tau))d\xi \\ &+ \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4kt}}}{\sqrt{4\pi kt}}\, \varphi(\xi)d\xi \end{aligned}$$

for all $x \in \mathbb{R}$ and $t \in]0,T]$.

Here, we consider the Banach space $(\Omega, \|\cdot\|)$, where

$$\Omega = \{v(x,t) : v, v_x \in C(\mathbb{R} \times [0,T]) \text{ and } \|v\| < +\infty\}$$

and

$$\|v\| = \sup_{x\in\mathbb{R},t\in[0,T]} |v(x,t)| + \sup_{x\in\mathbb{R},t\in[0,T]} |v_x(x,t)|.$$

Clearly $(\Omega, \|\cdot\|)$ with the metric d given by

$$d(u,v) = \sup_{x\in\mathbb{R},t\in[0,T]} |u(x,t) - v(x,t)| + \sup_{x\in\mathbb{R},t\in[0,T]} |u_x(x,t) - v_x(x,t)|$$

is a complete metric space.
Also the set Ω can be naturally endowed with the partial order:

$$\text{for all } u,v \in \Omega,\ \ u \preceq v \iff u(x,t) \le v(x,t) \text{ for any } x \in \mathbb{R} \text{ and } t \in [0,T].$$

Now we consider a monotone nondecreasing sequence $\{v_n\} \subseteq \Omega$ converging to $v \in \Omega$, for all $x \in \mathbb{R}$ and $t \in [0,T]$. This means that

$$v_1(x,t) \leq v_2(x,t) \leq v_3(x,t) \leq \cdots v_n(x,t) \leq \cdots \leq v(x,t)$$

for all $x \in \mathbb{R}$ and $t \in [0,T]$. Therefore, condition (iv) of Theorem 1.14 holds true, by choosing the set $\mathcal{M}$ as in Example 1.1.

Our theorem in this section links the existence of a solution for the initial-value problem (1.16) to the existence of a fixed point for an integral operator. The reader is referred to the paper of Aronson and Serrin [15] for further discussion of hypotheses below.

Theorem 1.15 *Assume that the following conditions hold:*

(a) for any $c > 0$ with $|u| < c$, the function $q(x,t,u)$ is bounded and uniformly Hölder continuous in x and in t for each compact subset of $\mathbb{R} \times [0,T]$;

(b) there exists a constant $c_q \leq (T + 2\pi^{-\frac{1}{2}}T^{\frac{1}{2}})^{-1}$ such that

$$0 \leq q(x,t,u(x,t)) - q(x,t,v(x,t)) \leq c_q \left[1 - e^{-\min\{d(u,v),1\}}\right]$$

for all $(u,v) \in \mathbb{R} \times \mathbb{R}$ with $v \preceq u$;

(c) there exists $u_0 \in \Omega$ such that

$$\begin{aligned} u_0(x,t) \quad &\leq \quad \int_0^t d\tau \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4k(t-\tau)}}}{\sqrt{4\pi k(t-\tau)}} q(\xi,\tau,u_0(\xi,\tau))d\xi \\ &+ \quad \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4k(t-\tau)}}}{\sqrt{4\pi kt}} \varphi(\xi)d\xi(x,t) \end{aligned}$$

for all $x \in \mathbb{R}$ and $t \in (0,T]$.

Then the initial-value problem (1.16) has at least a solution.

Proof. As said above, the initial-value problem (1.16) is equivalent to the integral equation (1.17) for all $x \in \mathbb{R}$ and $t \in]0,T]$. Then, the initial-value problem (1.16) possesses a solution if and only if the integral equation (1.17) has a solution u satisfying certain properties. Roughly speaking, this solution can be seen as fixed point of an integral operator, so that we can apply our fixed point theorems to get it.

To this aim, we define the operator $f : \Omega \to \Omega$ by:

$$\begin{aligned}(fu)(x,t) &= \int_0^t d\tau \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4k(t-\tau)}}}{\sqrt{4\pi k(t-\tau)}}\, q(\xi,\tau,u(\xi,\tau))d\xi \\ &+ \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4kt}}}{\sqrt{4\pi kt}}\, \varphi(\xi)d\xi\end{aligned}$$

for all $x \in \mathbb{R}$ and $t \in [0,T]$.

We will show that f satisfies all the requirements of Theorem 1.14, by choosing the set $\mathcal{M}$ as in Example 1.1. We have already remarked at the beginning of this section that condition (iv) of Theorem 1.14 holds true. Now, by condition (b) we deduce trivially that the operator f is nondecreasing. In fact, for all $u, v \in \Omega$ with $v \preceq u$, from

$$q(x,t,u(x,t)) \geq q(x,t,v(x,t))$$

for all $x \in \mathbb{R}$ and $t \in]0,T]$, we obtain

$$\begin{aligned}(fu)(x,t) &= \int_0^t d\tau \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4k(t-\tau)}}}{\sqrt{4\pi k(t-\tau)}}\, q(\xi,\tau,u(\xi,\tau))d\xi \\ &+ \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4kt}}}{\sqrt{4\pi kt}}\, \varphi(\xi)d\xi \\ &\geq \int_0^t d\tau \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4k(t-\tau)}}}{\sqrt{4\pi k(t-\tau)}}\, q(\xi,\tau,v(\xi,\tau))d\xi \\ &+ \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4kt}}}{\sqrt{4\pi kt}}\, \varphi(\xi)d\xi = (fv)(x,t)\end{aligned}$$

for all $x \in \mathbb{R}$ and $t \in]0,T]$.

Then, f is nondecreasing and, in view of Example 1.1, this implies that $\mathcal{M}$ is f-invariant, that is, condition (iii) of Theorem 1.14 holds true. Clearly, from assertion (c), $u_0 \preceq fu_0$ and hence condition (ii) of Theorem 1.14 holds true. Now we only need to show that f satisfies the contractive condition in Theorem 1.14. In fact, we get

$$
\begin{aligned}
& |(fu)(x,t)-(fv)(x,t)| \le \\
\le \quad & \int_0^t d\tau \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4k(t-\tau)}}}{\sqrt{4\pi k(t-\tau)}} |q(\xi,\tau,u(\xi,\tau)) - q(\xi,\tau,v(\xi,\tau))| d\xi \\
\le \quad & c_q \left[1 - e^{-\min\{d(u,v),1\}}\right] \int_0^t d\tau \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4k(t-\tau)}}}{\sqrt{4\pi k(t-\tau)}} d\xi \\
\le \quad & c_q \left[1 - e^{-\min\{d(u,v),1\}}\right] \cdot T
\end{aligned}
$$

for all $x \in \mathbb{R}$ and $t \in]0,T]$.
Analogous reasoning shows that

$$
\begin{aligned}
& \left| \frac{\partial (fu)(x,t)}{\partial x} - \frac{\partial (fv)(x,t)}{\partial x} \right| \le \\
\le \quad & c_q \left[1 - e^{-\min\{d(u,v),1\}}\right] \int_0^t d\tau \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} \right) d\xi \\
\le \quad & c_q \left[1 - e^{-\min\{d(u,v),1\}}\right] \cdot 2\pi^{-\frac{1}{2}} T^{\frac{1}{2}}.
\end{aligned}
$$

By combining the obtained results, we deduce that

$$
d(fu,fv) \le c_q(T + 2\pi^{-\frac{1}{2}} T^{\frac{1}{2}})(1 - e^{-\min\{d(u,v),1\}}) \le 1 - e^{-\min\{d(u,v),1\}},
$$

which further gives us

$$
d(fu,fv) \le \frac{1 - e^{-\min\{d(u,v),1\}}}{\min\{d(u,v),1\}} d(u,v). \tag{1.17}
$$

Therefore, condition (i) of Theorem 1.14 holds true with $\psi(s) = s$, and $\beta(s) = \frac{e^{-\min\{s,1\}} - 1}{-\min\{s,1\}}$ for $s > 0$ and $\beta(0) = \frac{1}{2}$. Thus, we can apply Theorem 1.14 to conclude that f has a fixed point and hence the initial-value problem (1.16) has a solution. □

Chapter 2

Fixed Points on Generalized Metric Spaces

Usefulness of metric spaces, in solving various practical problems, motivated many researchers to study their generalizations and extensions. Here, we consider the so called generalized metric spaces. We use this setting to provide new coincidence and common fixed point theorems. Also, to obtain these results, we use weak contractive conditions and a partial order.

2.1 Generalized Metric Spaces

Let $\mathbb{R}_+$ denote the set of all positive real numbers and $\mathbb{N}$ denote the set of all positive integers.

Definition 2.1 ([20]) *Let X be a non-empty set and $d : X \times X \to [0, +\infty[$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$ each of them different from x and y, one has*

(i) $d(x, y) = 0$ *if and only if* $x = y$*,*

(ii) $d(x, y) = d(y, x)$*,*

(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ *(rectangular inequality).*

Then (X, d) is called a generalized metric space (or shortly GMS).

We note that (iii) of Definition 2.1 does not ensure that d is continuous in each variable, see [84].

Definition 2.2 *Let (X, d) be a GMS, $\{x_n\}$ be a sequence in X and $x \in X$. Then*

(i) We say that $\{x_n\}$ is GMS convergent to x if and only if $d(x_n, x) \to 0$ as $n \to +\infty$. We denote this by $x_n \to x$.

(ii) We say that $\{x_n\}$ is a GMS Cauchy sequence if and only if for each $\varepsilon > 0$ there exists a natural number $n(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n > m \geq n(\varepsilon)$.

(iii) (X, d) is called GMS complete if every GMS Cauchy sequence is GMS convergent in X.

We note that a convergent sequence in a GMS is not necessarily a Cauchy sequence, see again [84].

2.2 Existing Fixed Point Results

First we define some class of functions which will be used to give new contraction condictions.

Definition 2.3 *We denote by Ψ the set of functions $\psi : [0, +\infty[\to [0, +\infty[$ satisfying the following hypotheses:*

(ψ1) ψ is continuous and nondecreasing;

(ψ2) $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.4 *We denote by Φ the set of functions $\varphi : [0, +\infty[\to [0, +\infty[$ satisfying the following hypotheses:*

(φ1) φ is lower semi-continuous;

(φ2) $\varphi(t) = 0$ if and only if $t = 0$.

In [57], Lakzian and Samet established the following fixed point theorem involving a pair of altering distance functions in a generalized complete metric space.

Theorem 2.1 *Let (X, d) be a Hausdorff and complete GMS and let $T : X \to X$ be a self-mapping satisfying*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y))$$

for all $x, y \in X$, where $\psi \in \Psi$ and $\varphi : [0, +\infty[\to [0, +\infty[$ is continuous and $\varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Definition 2.5 *Let X be a non-empty set and $T, f : X \to X$.*

(i) *A point $x \in X$ is said to be a common fixed point of T and f if $x = Tx = fx$.*

(ii) *A point $y \in X$ is called point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$. The point $x \in X$ such that $Tx = fx$ is said coincidence point.*

(iii) *The mappings T, f are said to be weakly compatible if they commute at their coincidence points (i.e. $Tfx = fTx$ whenever $Tx = fx$).*

In [34], Di Bari and Vetro established the following fixed point theorem.

Theorem 2.2 *Let (X, d) be a Hausdorff GMS and let T and f be self-mappings on X such that $TX \subset fX$. Assume that (fX, d) is a complete GMS and that the following condition holds:*

$$\psi(d(Tx, Ty)) \leq \psi(d(fx, fy)) - \varphi(d(fx, fy))$$

for all $x, y \in X$, where $\psi \in \Psi$ and $\varphi \in \Phi$. Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Lemma 2.1 *Let X be a non-empty set. Suppose that the mappings $T, f : X \to X$ have a unique point of coincidence $t \in X$. If T and f are weakly compatible, then T and f have a unique common fixed point.*

Proof. Let $t \in X$ be the point of coincidence of T and f, that is:

$$Tz = fz = t,$$

for some $z \in X$. Since T and f are weakly compatible, we observe that

$$Tz = fz \Rightarrow Tfz = fTz \Leftrightarrow Tt = ft.$$

But T and f have a unique point of coincidence, then $ft = t$. Hence we have $Tt = ft = t$. □

2.3 New Fixed Point Theorems

In this section, we prove some common fixed point results for two mappings satisfying an α-ψ-φ-contractive condition. For the notion of α-ψ-contractive type mappings, see Samet et al. [86]. Following [86], we introduce the notion of f-α-admissible mapping.

Definition 2.6 *Let (X,d) be a GMS, $T,f : X \to X$ and $\alpha : X \times X \to [0,+\infty[$. The mapping T is f-α-admissible if for all $x,y \in X$ such that $\alpha(fx,fy) \geq 1$ we have $\alpha(Tx,Ty) \geq 1$. If f is the identity mapping, then T is called α-admissible.*

Definition 2.7 *Let (X,d) be a GMS and $\alpha : X \times X \to [0,+\infty[$. X is α-regular if for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$, then we have $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N}$.*

Theorem 2.3 ([60] Theorem 3) *Let (X,d) be a GMS and let $T,f : X \to X$ be mappings such that $TX \subset fX$ and $\alpha : X \times X \to [0,+\infty[$. Assume that (fX,d) is a complete GMS and that the following condition holds:*

$$\psi(\alpha(fx,fy)d(Tx,Ty)) \leq \psi(M(x,y)) - \varphi(M(x,y)) \tag{2.1}$$

for all $x,y \in X$, where $\psi \in \Psi$, $\varphi \in \Phi$ and

$$M(x,y) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty)\}.$$

Assume also that the following conditions hold:

(i) T is f-α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(fx_0,Tx_0) \geq 1$;

(iii) X is α-regular and for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n,x_{n+1}) \geq 1$ we have $\alpha(x_m,x_n) \geq 1$ for all $m,n \in \mathbb{N}$ with $m < n$;

(iv) either $\alpha(fu,fv) \geq 1$ or $\alpha(fv,fu) \geq 1$ whenever $fu = Tu$ and $fv = Tv$.

Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(fx_0, Tx_0) \geq 1$. Define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_n = fx_{n+1} = Tx_n, \quad n = 0, 1, \ldots .$$

Moreover, we assume that if $y_n = Tx_n = Tx_{n+p} = y_{n+p}$, then we choose $x_{n+p+1} = x_{n+1}$. This can be done, since $TX \subset fX$. In particular, if $y_n = y_{n+1}$, then y_{n+1} is a point of coincidence of T and f. Consequently, we can suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$.

By condition (ii), we have $\alpha(fx_0, Tx_0) = \alpha(fx_0, fx_1) \geq 1$. Since, by hypothesis, T is f-α-admissible, we obtain

$$\alpha(Tx_0, Tx_1) = \alpha(fx_1, fx_2) \geq 1, \quad \alpha(Tx_1, Tx_2) = \alpha(fx_2, fx_3) \geq 1.$$

By induction, we get

$$\alpha(fx_n, fx_{n+1}) \geq 1 \qquad \text{for all } n \in \mathbb{N} \cup \{0\}\,.$$

Now, by (2.1), we have

$$\begin{aligned} \psi(d(Tx_n, Tx_{n+1})) &\leq \psi(\alpha(fx_n, fx_{n+1})\, d(Tx_n, Tx_{n+1})) \\ &\leq \psi(M(x_n, x_{n+1})) - \varphi(M(x_n, x_{n+1})) \end{aligned}$$

where

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1})\} \\ &= \max\{d(y_{n-1}, y_n), d(y_n . y_{n+1})\}\,. \end{aligned}$$

This implies

$$\psi(d(Tx_n, Tx_{n+1})) \leq \psi(M(x_n, x_{n+1})) - \varphi(M(x_n, x_{n+1})) \tag{2.2}$$

for all $n \in \mathbb{N}$. Now, if $M(x_n, x_{n+1}) = d(y_n, y_{n+1})$, from (2.2) we deduce

$$\psi(d(y_n, y_{n+1}) \leq \psi(d(y_n, y_{n+1})) - \varphi(d(y_n, y_{n+1}))$$

and hence $d(y_n, y_{n+1}) = 0$. If $M(x_n, x_{n+1}) = d(y_{n-1}, y_n) > 0$, then from (2.2) we get

$$\begin{aligned} \psi(d(y_n, y_{n+1}) &\leq \psi(d(y_{n-1}, y_n)) - \varphi(d(y_{n-1}, y_n)) \\ &< \psi(d(y_{n-1}, y_n)). \end{aligned} \tag{2.3}$$

Since ψ is non-decreasing, then $d(y_n, y_{n+1}) < d(y_{n-1}, y_n)$ for all $n \in \mathbb{N}$, that is, the sequence of nonnegative numbers $\{d(y_n, y_{n+1})\}$ is decreasing.

Hence, it converges to a nonnegative number, say $s \geq 0$. If $s > 0$, then letting $n \to +\infty$ in (2.3), we obtain $\psi(s) \leq \psi(s) - \varphi(s)$ which implies $s = 0$, that is

$$\lim_{n\to+\infty} d(y_n, y_{n+1}) = 0. \tag{2.4}$$

Suppose that $y_n \neq y_m$ for all $m \neq n$ and prove that $\{y_n\}$ is a GMS Cauchy sequence. First, we show that the sequence $\{d(y_n, y_{n+2})\}$ is bounded. Since $d(y_n, y_{n+1}) \to 0$, there exists $L > 0$ such that $d(y_n, y_{n+1}) \leq L$ for all $n \in \mathbb{N}$. If $d(y_n, y_{n+2}) > L$ for all $n \in \mathbb{N}$, from

$$\begin{aligned} M(x_n, x_{n+2}) &= \max\{d(fx_n, fx_{n+2}), d(fx_n, Tx_n), d(fx_{n+2}, Tx_{n+2})\} \\ &= d(y_{n-1}, y_{n+1}) \end{aligned}$$

and (iii) follows

$$\begin{aligned} \psi(d(y_n, y_{n+2})) &= \psi(d(Tx_n, Tx_{n+2})) \\ &\leq \psi(\alpha(fx_n, fx_{n+2})d(Tx_n, Tx_{n+2})) \\ &\leq \psi(M(x_n, x_{n+2})) - \varphi(M(x_n, x_{n+2})) \\ &< \psi(d(y_{n-1}, y_{n+1})). \end{aligned}$$

Thus the sequence $\{d(y_n, y_{n+2})\}$ is decreasing and hence is bounded. If for some $n \in \mathbb{N}$ we have $d(y_{n-1}, y_{n+1}) \leq L$ and $d(y_n, y_{n+2}) > L$, then from

$$\begin{aligned} \psi(d(y_n, y_{n+2})) &= \psi(d(Tx_n, Tx_{n+2})) \\ &\leq \psi(\alpha(fx_n, fx_{n+2})d(Tx_n, Tx_{n+2})) \\ &\leq \psi(M(x_n, x_{n+2})) - \varphi(M(x_n, x_{n+2})) \\ &< \psi(M(x_n, x_{n+2})) \leq \psi(L), \end{aligned}$$

we get $d(y_n, y_{n+2}) < L$, a contradiction. Then

$$d(y_n, y_{n+2}) > L \quad \text{or} \quad d(y_n, y_{n+2}) \leq L$$

for all $n \in \mathbb{N}$ and in both cases the sequence $\{d(y_n, y_{n+2})\}$ is bounded. Now, if

$$\lim_{n\to+\infty} d(y_n, y_{n+2}) = 0 \tag{2.5}$$

does not hold, then there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $d(y_{n_k}, y_{n_k+2}) \to s > 0$. From

$$d(y_{n_k-1}, y_{n_k+1}) \leq d(y_{n_k-1}, y_{n_k}) + d(y_{n_k}, y_{n_k+2}) + d(y_{n_k+1}, y_{n_k+2})$$

and

$$d(y_{n_k}, y_{n_k+2}) \leq d(y_{n_k-1}, y_{n_k}) + d(y_{n_k-1}, y_{n_k+1}) + d(y_{n_k+1}, y_{n_k+2})$$

we deduce that

$$\lim_{n\to+\infty} d(y_{n_k-1}, y_{n_k+1}) = s.$$

Now, by (2.1) with $x = x_{n_k}$ and $y = x_{n_k+2}$, we have

$$\begin{aligned} \psi(d(Tx_{n_k}, Tx_{n_k+2})) &\leq \psi(\alpha(fx_{n_k}, fx_{n_k+2})d(Tx_{n_k}, Tx_{n_k+2})) \\ &\leq \psi(M(x_{n_k}, x_{n_k+2})) - \varphi(M(x_{n_k}, x_{n_k+2})) \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} M(x_{n_k}, x_{n_k+2}) &= \max\{d(fx_{n_k}, fx_{n_k+2}), d(fx_{n_k}, Tx_{n_k}), d(fx_{n_k+2}, Tx_{n_k+2})\} \\ &= \max\{d(y_{n_k-1}, y_{n_k+1}), d(y_{n_k-1}, y_{n_k}), d(y_{n_k+1}, y_{n_k+2})\}. \end{aligned}$$

This implies

$$\lim_{k\to+\infty} M(x_{n_k}, x_{n_k+2}) = s.$$

From (2.6), as $k \to +\infty$, we get $\psi(s) \leq \psi(s) - \varphi(s)$ which implies $s = 0$.

Now, if possible, let $\{y_n\}$ be not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ with $n_k > m_k \geq k$ such that

$$d(y_{m_k}, y_{n_k}) \geq \varepsilon. \tag{2.7}$$

Further, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k - m_k \geq 3$ and satisfying (2.7). Then

$$d(y_{m_k}, y_{n_k-1}) < \varepsilon. \tag{2.8}$$

Now, using (2.7), (2.8) and the rectangular inequality, we get

$$\begin{aligned} \varepsilon &\leq d(y_{m_k}, y_{n_k}) \\ &\leq d(y_{n_k}, y_{n_k-2}) + d(y_{n_k-2}, y_{n_k-1}) + d(y_{n_k-1}, y_{m_k}) \\ &< d(y_{n_k}, y_{n_k-2}) + d(y_{n_k-2}, y_{n_k-1}) + \varepsilon. \end{aligned}$$

Letting $k \to +\infty$ in the above inequality, using (2.4) and (2.5), we obtain

$$d(y_{m_k}, y_{n_k}) \to \varepsilon^+. \tag{2.9}$$

From

$$\begin{aligned} d(y_{m_k}, y_{n_k}) &- d(y_{m_k-1}, y_{m_k}) - d(y_{n_k-1}, y_{n_k}) \\ &\leq d(y_{n_k-1}, y_{m_k-1}) \\ &\leq d(y_{n_k-1}, y_{n_k}) + d(y_{m_k}, y_{n_k}) + d(y_{m_k-1}, y_{m_k}), \end{aligned}$$

letting $k \to +\infty$, we obtain

$$d(y_{m_k-1}, y_{n_k-1}) \to \varepsilon. \tag{2.10}$$

From (2.1) with $x = x_{n_k}$ and $y = x_{m_k}$, we get

$$\begin{aligned} \psi(d(Tx_{m_k}, Tx_{n_k})) &\leq \psi(\alpha(fx_{m_k}, fx_{n_k})d(Tx_{m_k}, Tx_{n_k})) \\ &\leq \psi(M(fx_{m_k}, fx_{n_k})) - \varphi(M(fx_{m_k}, fx_{n_k})) \end{aligned}$$

where

$$\begin{aligned} M(fx_{m_k}, fx_{n_k}) &= \max\{d(fx_{m_k}, fx_{n_k}), d(fx_{n_k}, Tx_{n_k}), d(fx_{m_k}, Tx_{m_k})\} \\ &= \max\{d(y_{n_k-1}, y_{m_k-1}), d(y_{n_k-1}, y_{n_k}), d(y_{m_k-1}, y_{m_k})\}. \end{aligned}$$

Now, using the continuity of ψ and the lower semi-continuity of φ, as $k \to +\infty$ we obtain

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon),$$

which implies that $\varepsilon = 0$, a contradiction with $\varepsilon > 0$. Hence, $\{y_n\}$ is a GMS Cauchy sequence. Since (fX, d) is GMS complete, there exists $z \in fX$ such that $y_n \to z$. Let $y \in X$ be such that $fy = z$.

Since X is α-regular there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\alpha(y_{n_k-1}, fy) \geq 1$ for all $n \in \mathbb{N}$. If $fy \neq Ty$, applying inequality (2.1) with $x = x_{n_k}$, we obtain

$$\begin{aligned} \psi(d(Tx_{n_k}, Ty)) &\leq \psi(\alpha(fx_{n_k}, fy)d(Tx_{n_k}, Ty)) \\ &\leq \psi(M(fx_{n_k}, fy)) - \varphi(M(fx_{n_k}, fy)) \end{aligned}$$

where

$$\begin{aligned} M(fx_{n_k}, fy) &= \max\{d(fx_{n_k}, fy), d(fx_{n_k}, Tx_{n_k}), d(fy, Ty)\} \\ &= \max\{d(y_{n_k-1}, fy), d(y_{n_k-1}, y_{n_k}), d(fy, Ty)\}. \end{aligned}$$

Now, from

$$d(y_{n_k-1}, fy), d(y_{n_k-1}, y_{n_k}) \to 0 \quad \text{as } k \to +\infty$$

for n great enough we deduce $M(fx_{n_k}, fy) = d(fy, Ty)$. On the other hand,

$$d(fy, Ty) \leq d(fy, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k}) + d(Tx_{n_k}, Ty)$$

implies

$$d(fy, Ty) \leq \liminf_{k\to+\infty} d(Tx_{n_k}, Ty).$$

Since ψ is continuous and nondecreasing, for n great enough we get

$$\psi(d(fy, Ty)) \leq \liminf_{k\to+\infty} \psi(d(Tx_{n_k}, Ty)) \leq \psi(d(fy, Ty)) - \varphi(d(fy, Ty))$$

which implies $d(fy, Ty) = 0$, that is, $z = fy = Ty$ and so z is a point of coincidence for T and f.

Suppose that there exist $n, p \in \mathbb{N}$ such that $y_n = y_{n+p}$. We prove that $p = 1$, then $fx_{n+1} = Tx_n = Tx_{n+1} = y_{n+1}$ and so y_{n+1} is a point of coincidence of T and f. Assume $p > 1$, this implies that $d(y_{n+p-1}, y_{n+p}) > 0$. Using (2.3), we obtain

$$\begin{aligned}
\psi(d(y_n, y_{n+1})) &= \psi(d(y_{n+p}, y_{n+p+1})) \\
&\leq \psi(d(y_{n+p-1}, y_{n+p})) - \varphi(d(y_{n+p-1}, y_{n+p})) \\
&< \psi(d(y_{n+p-1}, y_{n+p})).
\end{aligned}$$

Since the sequence $d(y_n, y_{n+1})$ is decreasing, we deduce

$$\psi(d(y_n, y_{n+1})) < \psi(d(y_n, y_{n+1})),$$

a contradiction and hence $p = 1$. We deduce that T and f have a point of coincidence. The uniqueness of the point of coincidence is a consequence of the conditions (2.1) and (iv).

Now, if z is the point of coincidence of T and f, as T and f are weakly compatible, we deduce that $fz = Tz$ and so $z = fz = Tz$. Consequently z is the unique common fixed point of T and f. □

From Theorem 2.3, if we choose $f = I_X$ the identity mapping on X, we deduce the following corollary.

Corollary 2.1 *Let (X, d) be a complete GMS, let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, +\infty[$. Assume that the following condition holds:*

$$\psi(\alpha(x, y)d(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)) \tag{2.11}$$

for all $x, y \in X$, where $\psi \in \Psi$, $\varphi \in \Phi$ and

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Assume also that the following conditions hold:

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *X is α-regular and for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ we have $\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$;*

(iv) *either $\alpha(u, v) \geq 1$ or $\alpha(v, u) \geq 1$ whenever $u = Tu$ and $v = Tv$.*

Then T has a unique fixed point.

From Theorem 2.3, if the function $\alpha : X \times X \to [0, +\infty[$ is such that $\alpha(x, y) = 1$ for all $x, y \in X$, we deduce the following theorem.

Theorem 2.4 ([60] Theorem 4) *Let (X, d) be a GMS and let $T, f : X \to X$ be mappings such that $TX \subset fX$. Assume that (fX, d) is a complete GMS and that the following condition holds:*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)) \tag{2.12}$$

for all $x, y \in X$, where $\psi \in \Psi$, $\varphi \in \Phi$ and

$$M(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\}.$$

Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Let X be a non-empty set. If (X, d) is a GMS and $(X, \preceq)$ is a partially ordered set, then $(X, d, \preceq)$ is called a partially ordered GMS. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. Let $(X, \preceq)$ be a partially ordered set and $T, f : X \to X$ be two mappings. T is called an f-nondecreasing mapping if $Tx \preceq Ty$ whenever $fx \preceq fy$ for all $x, y \in X$.

From Theorem 2.3, in the setting of partially ordered GMS spaces we obtain the following theorem.

Theorem 2.5 ([60] Theorem 5) *Let $(X, d, \preceq)$ be a partially ordered GMS and let $T, f : X \to X$ be mappings such that $TX \subset fX$. Assume that (fX, d) is a complete GMS and that the following condition holds:*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)) \tag{2.13}$$

for all $x, y \in X$ such that $fx \preceq fy$, where $\psi \in \Psi$, $\varphi \in \Phi$ and

$$M(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\}.$$

Assume also that the following conditions hold:

(i) T *is* f*-nondecreasing;*

(ii) there exists $x_0 \in X$ *such that* $fx_0 \preceq Tx_0$*;*

(iii) if $\{x_n\} \subset X$ *is such that* $x_n \preceq x_{n+1}$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x$*, then we have* $x_n \preceq x$ *for all* $n \in \mathbb{N}$*;*

(iv) for all $u, v \in X$ *such that* $fu = Tu$ *and* $fv = Tv$*, then* fu *and* fv *are comparable.*

Then T *and* f *have a unique point of coincidence in* X*. Moreover, if* T *and* f *are weakly compatible, then* T *and* f *have a unique common fixed point.*

Proof. Define the mapping $\alpha : X \times X \to [0, +\infty[$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in fX \quad \text{and} \quad x \preceq y \\ 0 & \text{otherwise.} \end{cases}$$

The reader can show easily that T is an f-α-admissible mapping. Now, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to +\infty$. By the definition of α, we have $x_n, x_{n+1} \in fX$ and $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. Since fX is complete, we deduce that $x \in fX$. By (iii), $x_n \preceq x$ for all $n \in \mathbb{N}$ and so $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. The same considerations show that (ii) and (iv) of this theorem imply (ii) and (iv) of Theorem 2.3. Thus the hypotheses (i)-(iv) of Theorem 2.3 are satisfied. Also the contractive condition (2.1) is satisfied, since $\alpha(fx, fy) = 1$ for all $x, y \in X$ such that $fx \preceq fy$. Otherwise $\psi(\alpha(fx, fy)d(Tx, Ty)) = 0$ and so condition (1) holds. By Theorem 2.3, T and f have a unique common fixed point. $\square$

Now, from Theorem 2.3, we can derive many interesting fixed point results in generalized metric spaces. Denote by Λ the set of functions $\gamma : [0, +\infty[\to [0, +\infty[$ Lebesgue integrable on each compact subset of $[0, +\infty[$ such that for every $\varepsilon > 0$, we have

$$\int_0^\varepsilon \gamma(s)ds > 0.$$

As the function $\psi : [0, +\infty[\to [0, +\infty[$ defined by $\psi(t) = \int_0^t \gamma(s)ds$ belongs to Ψ, we obtain the following theorem.

Theorem 2.6 ([60] Theorem 6) *Let (X, d) be a GMS and let $T, f : X \to X$ be mappings such that $TX \subset fX$ and $\alpha : X \times X \to [0, +\infty[$. Assume that (fX, d) is a complete GMS and that the following condition holds:*

$$\int_0^{\alpha(fx,fy)d(Tx,Ty)} \gamma(s)ds \leq \int_0^{M(x,y)} \gamma(s)ds - \int_0^{M(x,y)} \delta(s)ds$$

for all $x, y \in X$, where $\gamma, \delta \in \Lambda$ and

$$M(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\}.$$

Assume also that the following conditions hold:

(i) T is f-α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(fx_0, Tx_0) \geq 1$;

(iii) X is α-regular and for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ we have $\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$;

(iv) either $\alpha(fu, fv) \geq 1$ or $\alpha(fv, fu) \geq 1$ whenever $fu = Tu$ and $fv = Tv$.

Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Taking $\delta(s) = (1 - k)\gamma(s)$ for $k \in [0, 1[$ in Theorem 2.6, we obtain the following result.

Theorem 2.7 ([60] Theorem 7) *Let (X, d) be a GMS and let $T, f : X \to X$ be mappings such that $TX \subset fX$ and $\alpha : X \times X \to [0, +\infty[$. Assume that (fX, d) is a complete GMS and that the following condition holds:*

$$\int_0^{\alpha(fx,fy)d(Tx,Ty)} \gamma(s)ds \leq k \int_0^{M(x,y)} \gamma(s)ds$$

for all $x, y \in X$, where $k \in [0, 1[$. Assume also that the following conditions hold:

(i) T is f-α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(fx_0, Tx_0) \geq 1$;

(iii) *X is α-regular and for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ we have $\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$;*

(iv) *either $\alpha(fu, fv) \geq 1$ or $\alpha(fv, fu) \geq 1$ whenever $fu = Tu$ and $fv = Tv$.*

Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Example 2.1 *Let $X = [1,2] \cup A$ with $A = \left\{\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right\}$. Define the generalized metric d on X as follows:*

$$d(x,y) = d(y,x), \quad d(x,y) = |x-y| \text{ if } \{x,y\} \cap [1,2] \neq \emptyset,$$

$$d\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{6} + 3a, \quad d\left(\frac{1}{2}, \frac{1}{4}\right) = \frac{1}{4} + 6a, \quad d\left(\frac{1}{2}, \frac{1}{5}\right) = \frac{3}{10} + 2a,$$

$$d\left(\frac{1}{3}, \frac{1}{4}\right) = \frac{1}{12} + 2a, \quad d\left(\frac{1}{3}, \frac{1}{5}\right) = \frac{2}{15} + 6a, \quad d\left(\frac{1}{4}, \frac{1}{5}\right) = \frac{1}{20} + 3a,$$

where $a = 1/24$.

Clearly, (X, d) is a complete GMS. Let $T : X \to X$ and $\psi, \varphi : [0, +\infty[\to [0, +\infty[$ defined by

$$Tx = \begin{cases} \frac{1}{4} & \text{if } x \in A \cup \left\{\frac{3}{2}\right\}, \\ 3 - x & \text{if } x \in [1,2] \setminus \left\{\frac{3}{2}\right\}, \end{cases} \qquad \psi(t) = t \text{ and } \varphi(t) = \frac{t}{5}.$$

Also consider $\alpha : X \times X \to [0, +\infty[$ given by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in A \text{ or } x = y \\ 0 & \text{otherwise.} \end{cases}$$

Then T and α satisfy all the conditions of Corollary 2.1 and hence T has a unique fixed point on X, that is, $x = 1/4$.

We note that if X is endowed with the standard metric $d(x,y) = |x-y|$ for all $x, y \in X$, then do not exist $\psi, \varphi : [0, +\infty[\to [0, +\infty[$, where $\psi \in \Psi$ and $\varphi \in \Phi$, such that

$$\psi(d(Tx, Ty)) \leq \psi(M(x,y)) - \varphi(M(x,y))$$

for all $x, y \in X$.

Chapter 3

Fixed Points on Partial Metric Spaces

As mentioned in the Introduction, partial metric spaces are interesting generalizations of metric spaces, in particular for applications in theoretical computer science. We use the partial metric setting to give new fixed and common fixed point theorems for single-valued, multi-valued and mixed multi-valued mappings.

3.1 Partial Metric Spaces

Matthews [63] introduced the notion of a partial metric as follows.

Definition 3.1 ([63]) *A partial metric on a nonempty set X is a function $p : X \times X \to [0, +\infty[$ such that, for all $x, y, z \in X$*
(p1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
(p2) $p(x, x) \leq p(x, y)$,
(p3) $p(x, y) = p(y, x)$,
(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x, y) = 0$, then from (p1) and (p2) $x = y$. But if $x = y$, $p(x, y)$ may not be 0.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where

$$B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$$

for all $x \in X$ and $\epsilon > 0$.

Definition 3.2 *Let (X, p) be a partial metric space. Then:*

1. *A sequence $\{x_n\}$ in X converges to a point $x \in X$, with respect to τ_p, if $\lim_{n \to\ +\infty} p(x, x_n) = p(x, x)$.*
   This will be denoted as $x_n \to x$, $n \to +\infty$ or $\lim_{n \to +\infty} x_n = x$.

2. *A sequence $\{x_n\}$ in X is called a Cauchy sequence if $\lim_{n,m \to +\infty} p(x_n, x_m)$ exists (and is finite).*

3. *The space (X, p) is said to be complete if every Cauchy sequence*

   $\{x_n\} \subset X$ converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m \to +\infty} p(x_n, x_m)$.

4. *A sequence $\{x_n\}$ in X is called 0-Cauchy if:*

   $\lim_{n,m \to +\infty} p(x_n, x_m) = 0$. The space (X, p) is said to be 0-complete if every 0-Cauchy sequence in X converges (in τ_p) to a point $x \in X$ such that $p(x, x) = 0$.

If p is a partial metric on X, then the function $p^s : X \times X \to [0, +\infty[$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

defines a metric on X.

Furthermore, a sequence $\{x_n\}$ converges in (X, p^s) to a point $x \in X$ if and only if

$$\lim_{n,m \to \infty} p(x_n, x_m) = \lim_{n \to \infty} p(x_n, x) = p(x, x).$$

Example 3.1 *Paradigmatic examples of a partial metric space are:*

- *The pair $([0, +\infty[, p)$, where $p : [0, +\infty[\times [0, +\infty[\to [0, +\infty[$ is defined by*

  $$p(x, y) = \max\{x, y\} \quad \textit{for all} \quad x, y \in [0, +\infty[.$$

  The corresponding metric on X is:

  $p^s(x, y) = 2 \max\{x, y\} - x - y = |x - y|$.

- *If* (X, d) *is a metric space and* $c \geq 0$ *is arbitrary, then* $p(x, y) = d(x, y) + c$ *defines a partial metric on* X *and the corresponding metric is* $p^s(x, y) = 2d(x, y)$.

- *Let* $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$ *and define:* $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$.

  Then (X, p) *is a partial metric space.*

Remark 3.1 ([61] Remark 2.4) *Clearly, a limit of a sequence in a partial metric space need not be unique. Moreover, the function* $p(\cdot, \cdot)$ *need not be continuous in the sense that* $x_n \to x$ *and* $y_n \to y$ *implies* $p(x_n, y_n) \to p(x, y)$. *For example, if* $X = [0, +\infty[$ *and* $p(x, y) = \max\{x, y\}$ *for* $x, y \in X$, *then for* $\{x_n\} = \{1\}$, $p(x_n, x) = x = p(x, x)$ *for each* $x \geq 1$ *and so, for example,* $x_n \to 2$ *and* $x_n \to 3$ *when* $n \to +\infty$.

Now, we recall the definition of partial Hausdorff metric and some properties that can be found in [1, 11]. Let $CB^p(X)$ be the family of all nonempty, closed and bounded subsets of the partial metric space (X, p), induced by the partial metric p. Note that closedness is taken from (X, τ_p) and boundedness is given as follows: A is a bounded subset in (X, p) if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0, M)$, that is, $p(x_0, a) < p(x_0, x_0) + M$.

For $A, B \in CB^p(X)$ and $x \in X$, define

$$\begin{aligned} p(x, A) &= \inf\{p(x, a),\ a \in A\},\ \delta_p(A, B) = \sup\{p(a, B) : a \in A\} \text{ and} \\ \delta_p(B, A) &= \sup\{p(b, A) : b \in B\}. \end{aligned}$$

Remark 3.2 ([7]) *Let* (X, p) *be a partial metric space and* A *any nonempty set in* (X, p), *then*

$$a \in \bar{A} \text{ if and only if } p(a, A) = p(a, a), \tag{3.1}$$

where $\bar{A}$ *denotes the closure of* A *with respect to the partial metric* p. *Note that* A *is closed in* (X, p) *if and only if* $A = \bar{A}$.

In the following proposition, we bring some properties of the mapping $\delta_p : CB^p(X) \times CB^p(X) \to [0, +\infty[$.

Proposition 3.1 ([11] Proposition 2.2) *Let* (X, p) *be a partial metric space. For any* $A, B, C \in CB^p(X)$, *we have the following:*

(i) $\delta_p(A, A) = \sup\{p(a, a) : a \in A\}$*;*

(ii) $\delta_p(A, A) \leq \delta_p(A, B)$*;*

(iii) $\delta_p(A, B) = 0$ *implies that* $A \subseteq B$*;*

(iv) $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)$*.*

Let (X, p) be a partial metric space. For $A, B \in CB^p(X)$, define

$$H_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\}. \tag{3.2}$$

In the following proposition, we bring some properties of the mapping H_p.

Proposition 3.2 ([11] Proposition 2.3) *Let* (X, p) *be a partial metric space. For all* $A, B, C \in CB^p(X)$*, we have*

(h1) $H_p(A, A) \leq H_p(A, B)$*;*

(h2) $H_p(A, B) = H_p(B, A)$*;*

(h3) $H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(c, c)$*.*

Corollary 3.1 ([11] Corollary 2.4) *Let* (X, p) *be a partial metric space. For* $A, B \in CB^p(X)$ *the following holds:*

$$H_p(A, B) = 0 \text{ implies that } A = B. \tag{3.3}$$

Remark 3.3 ([11] Remark 2.5) *The converse of Corollary 3.1 is not true in general as shown by the following example.*

Example 3.2 ([11] Example 2.6) *Let* $X = [0, 1]$ *be endowed with the partial metric* $p : X \times X \to [0, +\infty)$ *defined by*

$$p(x, y) = \max\{x, y\} \quad \text{for all } x, y \in X. \tag{3.4}$$

From (i) of Proposition 3.1, we have

$$H_p(X, X) = \delta_p(X, X) = \sup\{x : 0 \leq x \leq 1\} = 1 \neq 0. \tag{3.5}$$

In view of Proposition 3.2 and Corollary 3.1, we call the mapping

$$H_p : CB^p(X) \times CB^p(X) \to [0, +\infty[,$$

a partial Hausdorff metric induced by p.

Remark 3.4 ([11] Remark 2.7) *It is easy to show that any Hausdorff metric is a partial Hausdorff metric. The converse is not true (see Example 3.2).*

Lemma 3.1 ([63]) *Let (X, p) be a partial metric space. Then one has the following:*

(a) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).*

(b) *The space (X, p) is complete if and only if the metric space (X, p^s) is complete.*

Definition 3.3 *Let X be a nonempty set. Then, $(X, p, \preceq)$ is called an ordered partial metric space if:*

(i) *(X, p) is a partial metric space;*

(ii) *$(X, \preceq)$ is a partially ordered set.*

We give the following auxiliary lemmas which are useful to prove some fixed point theorems in a partial metric space.

Lemma 3.2 ([72] Lemma 2) *Let (X, p) be a partial metric space and $\{x_n\} \subset X$. If $x_n \to x \in X$ and $p(x, x) = 0$, then $\lim_{n\to+\infty} p(x_n, z) = p(x, z)$ for all $z \in X$.*

Lemma 3.3 ([37] Lemma 2.8) *Let (X, p) be a metric space and $\{x_n\}$ be a sequence in X such that:*

$$\lim_{n\to+\infty} p(x_{n+1}, x_n) = 0.$$

If $\{x_{2n}\}$ is not a Cauchy sequence in (X, p), then there exist $\epsilon > 0$ and two sequences $\{m_k\}$, $\{n_k\}$ of positive integers, with $m_k < n_k$, such that the following four sequences:

$$\{p(x_{2m_k}, x_{2n_k})\}, \{p(x_{2m_k}, x_{2n_k+1})\}, \{p(x_{2m_k-1}, x_{2n_k})\}, \{p(x_{2m_k-1}, x_{2n_k+1})\}$$

tend to ϵ as $k \to +\infty$.

Definition 3.4 ([61] Definition 3.2) *Let $f : X \to X$ and $\alpha : X \times X \to [0, +\infty[$. The mapping f is α-admissible if for all $x, y \in X$ such that $\alpha(x, y) \geq 1$, we have $\alpha(fx, fy) \geq 1$.*

Definition 3.5 ([61] Definition 3.3) *Let (X, p) be a partial metric space and let $\alpha : X \times X \to [0, +\infty[$. X is called α-regular if for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$.*

3.2 Fixed Points of Single-Valued Mappings

The following theorem is one of our main results.

Theorem 3.1 ([61] Theorem 3.5) *Let (X, p) be a complete partial metric space and let $\alpha : X \times X \to [0, +\infty[$ be a function. Let $f : X \to X$ be a mapping. Suppose that there exists $\beta \in S$ such that*

$$\alpha(x, fx)\alpha(y, fy)p(fx, fy) \leq \beta(M(x, y))M(x, y) \tag{3.6}$$

for all $x, y \in X$, where

$$M(x, y) = \max\left\{p(x, y), p(x, fx), p(y, fy),\ \frac{1}{2}[p(x, fy) + p(fx, y)]\right\}.$$

Assume also that the following conditions hold:

(i) f is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;

(iii) for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$, we have $\alpha(x, fx) \geq 1$;

(iv) $\alpha(x, fx) \geq 1$ for all $x \in Fix(f)$.

Then f has a unique fixed point $z \in X$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Define the sequence $\{x_n\} \subset X$ by

$$x_n = fx_{n-1} \quad \text{for all } n \in \mathbb{N}.$$

Since, by hypothesis, f is α-admissible, we obtain

$$\alpha(fx_0, fx_1) = \alpha(x_1, x_2) \geq 1 \quad \alpha(fx_1, fx_2) = \alpha(x_2, x_3) \geq 1.$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $x_n = x_{n+1} = fx_n$ and so x_n is a fixed point of f.
Now, we assume $p(x_{n+1}, x_n) > 0$ for each $n \in \mathbb{N} \cup \{0\}$. First, we will prove that the sequence $\{p(x_{n+1}, x_n)\}$ is decreasing and tends to 0 as $n \to +\infty$. By (3.6), for each $n \in \mathbb{N}$, we have:

$$\begin{aligned} p(x_{n+2}, x_{n+1}) &= p(fx_{n+1}, fx_n) \\ &\leq \alpha(x_{n+1}, fx_{n+1})\alpha(x_n, fx_n)p(fx_{n+1}, fx_n) \\ &\leq \beta(M(x_{n+1}, x_n))M(x_{n+1}, x_n) \\ &< M(x_{n+1}, x_n) \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} M(x_{n+1}, x_n) &= \max\{p(x_{n+1}, x_n), p(x_{n+1}, x_{n+2}), p(x_n, x_{n+1}), \\ &\quad \frac{1}{2}[p(x_{n+1}, x_{n+1}) + p(x_{n+2}, x_n)]\}. \end{aligned}$$

Since in a partial metric space we have

$$p(x_{n+1}, x_{n+1}) + p(x_{n+2}, x_n) \leq p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n),$$

then we get

$$M(x_{n+1}, x_n) = \max\{p(x_{n+1}, x_n), p(x_{n+2}, x_{n+1})\}.$$

If $M(x_{n+1}, x_n) = p(x_{n+2}, x_{n+1})$, by (3.7), we have a contradiction. Then

$$M(x_{n+1}, x_n) = p(x_{n+1}, x_n).$$

Again using (3.7) it follows $0 < p(x_{n+2}, x_{n+1}) < p(x_{n+1}, x_n)$.

Hence, the sequence $\{p(x_{n+1}, x_n)\}$ is decreasing and bounded from below, thus it converges to some $r \geq 0$. Suppose that $r > 0$. By (3.7), we have

$$\frac{p(x_{n+2}, x_{n+1})}{p(x_{n+1}, x_n)} \leq \beta(p(x_{n+1}, x_n)) \leq 1$$

for all $n \in \mathbb{N} \cup \{0\}$ which yields that

$$\lim_{n \to +\infty} \beta(p(x_{n+1}, x_n)) = 1.$$

On the other hand, since $\beta \in S$, we have $\lim_{n \to +\infty} p(x_{n+1}, x_n) = 0$ and so $r = 0$. In order to prove that $\{x_n\}$ is a Cauchy sequence in (X, p), suppose the contrary, that is, $\{x_n\}$ is not a Cauchy sequence. Using Lemma 3.3, we know that there exist $\epsilon > 0$ and two sequences $\{m_k\}$, $\{n_k\}$ of positive integers, with $m_k < n_k$, such that the following four sequences

$$\{p(x_{2m_k}, x_{2n_k})\}, \{p(x_{2m_k}, x_{2n_k+1})\}, \{p(x_{2m_k-1}, x_{2n_k})\}, \{p(x_{2m_k-1}, x_{2n_k+1})\}$$

tend to ϵ as $k \to +\infty$.
Putting, in the contractive condition (3.6), $x = x_{2m_k-1}$ and $y = x_{2n_k}$, it follows that:

$$\begin{aligned} & p(x_{2m_k}, x_{2n_k+1}) \qquad (3.8)\\ \leq\ & \alpha(x_{2m_k-1}, fx_{2m_k-1})\alpha(x_{2m_k}, fx_{2m_k})p(fx_{2m_k-1}, fx_{2m_k})\\ \leq\ & \beta(M(x_{2m_k-1}, x_{2n_k}))M(x_{2m_k-1}, x_{2n_k})\\ <\ & M(x_{2m_k-1}, x_{2n_k}),\end{aligned}$$

where

$$\begin{aligned} M(x_{2m_k-1}, x_{2n_k}) \ =\ & \max\{p(x_{2m_k-1}, x_{2n_k}), p(x_{2m_k-1}, x_{2m_k}), p(x_{2n_k}, x_{2n_k+1}),\\ & \frac{1}{2}[p(x_{2m_k-1}, x_{2n_k+1}) + p(x_{2m_k}, x_{2n_k})]\}.\end{aligned}$$

Letting $k \to +\infty$, we get $M(x_{2m_k-1}, x_{2n_k}) \to \epsilon$. From (3.8) we have

$$\frac{p(x_{2m_k}, x_{2n_k+1})}{M(x_{2m_k-1}, x_{2n_k})} \leq \beta(M(x_{2m_k-1}, x_{2n_k})) \leq 1,$$

for all $k \in \mathbb{N}$. From the previous inequality, as $k \to +\infty$, we obtain

$$\lim_{k\to+\infty} \beta(M(x_{2m_k-1}, x_{2n_k})) = 1.$$

Since $\beta \in S$, we have

$$\lim_{k\to+\infty} M(x_{2m_k-1}, x_{2n_k}) = 0,$$

which is a contradiction. This implies that $\epsilon = 0$. Therefore, $\{x_n\}$ is a Cauchy sequence in (X, p). Since (X, p) is complete, it follows that the sequence $\{x_n\}$ converges to some $z \in X$. We say

$$p(z, z) = \lim_{n\to+\infty} p(x_n, z) = \lim_{n,m\to+\infty} p(x_n, x_m) = 0. \tag{3.9}$$

Now, we show that z is a fixed point of f. If $p(z, fz) > 0$, using condition (iii) and (3.6) with $x = x_n$ and $y = z$, we get

$$\begin{aligned} p(x_{n+1}, fz) \ & \leq\ \alpha(x_n, fx_n)\alpha(z, fz)p(fx_{n_k}, fz)\\ & \leq\ \beta(M(x_n, z))M(x_n, z).\end{aligned}$$

Now, for n large enough we have $M(x_n, z) = p(z, fz)$ and so, from the previous inequality, $p(z, z) = 0$ and Lemma 3.2, we obtain

$$1 = \lim_{n\to+\infty} \frac{p(x_{n+1}, fz)}{M(x_n, z)} = \lim_{n\to+\infty} \beta(M(x_n, z)) \leq 1.$$

This implies

$$\lim_{n\to+\infty} M(x_n, z) = 0,$$

a contradiction. Thus, $p(z, fz) = 0$ and hence $fz = z$, that is, z is a fixed point of f. Assume that u and v, with $u \neq v$, are two fixed points of f. Then

$$\begin{aligned} 0 < p(u, v) &\leq \alpha(u, fu)\alpha(v, fv)p(fu, fv) \\ &\leq \beta(M(u, v))M(u, v) < M(u, v), \end{aligned}$$

where

$$\begin{aligned} M(u, v) &= \max\{p(u, v), p(u, fu), p(v, fv)), \\ &\quad \frac{1}{2}[p(u, fv)) + p(fu, v)]\} \\ &= p(u, v). \end{aligned}$$

It follows

$$0 < p(u, v) < M(u, v) = p(u, v),$$

a contradiction. Therefore, we get $u = v$ and this completes the proof.

□

The following theorem is another main result of the paper [61].

Theorem 3.2 ([61] Theorem 3.6) *Let (X, p) be a complete partial metric space and let $\alpha : X \times X \to [0, +\infty[$ be a function. Let $f : X \to X$ be a mapping. Suppose that there exists $\beta \in S$ such that*

$$\alpha(x, y)p(fx, fy) \leq \beta(M(x, y))M(x, y) \tag{3.10}$$

for all $x, y \in X$, where

$$M(x, y) = \max\left\{p(x, y), p(x, fx), p(y, fy),\ \frac{1}{2}[p(x, fy) + p(fx, y)]\right\}.$$

Assume also that the following conditions hold:

(i) f is α-admissible;

(ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq 1$*;*

(iii) X *is* α*-regular and for every sequence* $\{x_n\} \subset X$ *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *we have* $\alpha(x_m, x_n) \geq 1$ *for all* $m, n \in \mathbb{N}$ *with* $m < n$*;*

(iv) $\alpha(x, y) \geq 1$ *for all* $x, y \in Fix(f)$*.*

Then f *has a unique fixed point* $z \in X$*.*

Proof. Proceeding as in the proof of Theorem 3.1, we deduce that $\{x_n\}$ is a Cauchy sequence in (X, p) such that $p(x_{n+1}, x_n) \to 0$ as $n \to +\infty$. Since (X, p) is complete, it follows that the sequence $\{x_n\}$ converges to some $z \in X$ such that

$$p(z, z) = \lim_{n\to+\infty} p(x_n, z) = \lim_{n,m\to+\infty} p(x_n, x_m) = 0. \tag{3.11}$$

Now, we show that z is a fixed point of f. Since X is α-regular, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, z) \geq 1$ for all $k \in \mathbb{N}$. If $p(z, fz) > 0$, using condition (3.10) with $x = x_{n_k}$ and $y = z$, we get that

$$p(x_{n_k+1}, fz) \leq \alpha(x_{n_k}, z)p(fx_{n_k}, fz) \leq \beta(M(x_{n_k}, z))M(x_{n_k}, z).$$

Now, for k large enough we have $M(x_{n_k}, z) = p(z, fz)$ and so, from the previous inequality and Lemma 3.2, we obtain

$$1 = \lim_{k\to+\infty} \frac{p(x_{n_k+1}, fz)}{M(x_{n_k}, z)} = \lim_{k\to+\infty} \beta(M(x_{n_k}, z)) \leq 1.$$

This implies $\lim_{k\to+\infty} M(x_{n_k}, z) = 0$, a contradiction. Thus, $p(z, fz) = 0$ and hence $fz = z$, that is, z is a fixed point of f. Assume that u and v, with $u \neq v$, are two fixed points of f. Then

$$0 < p(u, v) \leq \alpha(u, v)p(fu, fv) \leq \beta(M(u, v))M(u, v) < M(u, v),$$

where $M(u, v) =$

$$\max\left\{p(u, v), p(u, fu), p(v, fv)), \frac{1}{2}[p(u, fv)) + p(fu, v)]\right\} = p(u, v).$$

It follows $0 < p(u, v) < M(u, v) = p(u, v)$, a contradiction. Therefore, we get $u = v$ and this completes the proof. $\square$

Example 3.3 Let $X = [0,1]$, $d(x,y) = |x-y|$ for all $x,y \in X$, $p(x,y) = \max\{x,y\}$ for all $x,y \in X$, $\beta(t) = \frac{e^{-t}}{t+1}$ for each $t > 0$ and $\beta(0) = \frac{1}{2}$. Let

$$\alpha(x,y) = \begin{cases} \frac{1}{4} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0). \end{cases}$$

The mapping $f : X \to X$ defined by $f(x) = \frac{x}{3}$ is α-admissible, but it does not satisfy the conditions of Geraghty's theorem in the metric space (X,d). Indeed, taking $x = 1$ and $y = 0$, we have

$$d(f1, f0) = d(\frac{1}{3}, 0) = \left|\frac{1}{3} - 0\right| = \frac{1}{3}$$

and

$$\beta(d(1,0))d(1,0) = \beta(|1-0|)\,|1-0| = \beta(1) = \frac{1}{2e}$$

Since $\frac{1}{3} > \frac{1}{2e}$ Geraghty's theorem cannot be used to prove the existence of a fixed point of f. Also we note that the mapping f does not satisfy the condition of Theorem 3.1 of [37] with respect to the partial metric defined above, because of

$$p(f1, f0) = \frac{1}{3} > \frac{1}{2e} = \beta(p(1,0))p(1,0).$$

On the other hand, taking $x,y \in X$ with, for example, $x \geq y$ and $x > 0$, then:

$$M(x,y) = \max\left\{p(x,y), p(x,fx), p(y,fy), \frac{1}{2}[p(x,fy) + p(fx,y)]\right\} = x$$

$$\alpha(x,y)p(fx,fy) = \frac{1}{12}x$$

and

$$\beta(M(x,y))M(x,y) = \beta(x)x = \frac{e^{-x}}{x+1}x.$$

Now, from $\frac{1}{12} < \frac{1}{2e} \leq \frac{e^{-x}}{(x+1)}$ for all $x \in [0,1]$, we get that (3.10) holds. Since the conditions (i)-(iv) of Theorem 3.2 are satisfied, then f has a unique fixed point ($z = 0$).

The existence of fixed points in partially ordered sets has been considered in [75]. Later on, some generalizations of [75] are given in [37, 67, 68, 69, 73, 80, 86]. Several applications of these results to matrix equations are presented in [75]. Moreover, some applications to periodic boundary value problems and to some particular problems are given, respectively, in [67, 68]. The following theorem ensures the existence of a fixed point for mappings in the setting of ordered partial metric spaces.

Theorem 3.3 ([61] Theorem 4.1) *Let $(X, p, \preceq)$ be a complete ordered partial metric space. Let $f : X \to X$ be a non-decreasing mapping. Suppose that there exists $\beta \in S$ such that*

$$p(fx, fy) \leq \beta(M(x, y))M(x, y) \tag{3.12}$$

for all $x, y \in X$ with $x \preceq y$, where

$$M(x, y) = \max \left\{ p(x, y), p(x, fx), p(y, fy), \frac{1}{2}[p(x, fy) + p(fx, y)] \right\}.$$

Assume also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;

(ii) X is such that, if a non-decreasing sequence $\{x_n\}$ converges to x, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$;

(iii) x, y are comparable whenever $x, y \in Fix(f)$.

Then f has a unique fixed point $z \in X$.

Proof. Define the mapping $\alpha : X \times X \to [0, +\infty[$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y \\ 0 & \text{otherwise} . \end{cases}$$

The reader can show easily that f is an α-admissible mapping and so (i) of Theorem 3.2 holds. The condition (i) above ensures that (ii) of Theorem 3.2 holds. Now, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to +\infty$. By the definition of α, we have $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. By (ii), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$ and so $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$ and hence X is α-regular. Further, $\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$. Hence (iii) of Theorem 3.2 holds. The same considerations show that (iii) of this theorem implies (iv) of Theorem 3.2. Thus the hypotheses (i)-(iv) of Theorem 3.2 are satisfied. Also the contractive condition (3.10) is satisfied, because of $\alpha(x, y) = 1$ for all $x, y \in X$ such that $x \preceq y$ and $\alpha(x, y) = 0$ if $x \not\preceq y$. Hence by Theorem 3.2, f have a unique fixed point. $\square$

3.3 Fixed Points of Mixed Multi-Valued Mappings

Very recently, Romaguera [77] introduced the concept of mixed multi-valued mappings, so that both a single-valued mapping $T : X \to X$ and a multi-valued mapping $T : X \to CB^p(X)$ (the family of all nonempty, closed and bounded subsets of a partial metric space X), are mixed multi-valued mappings. In this paper we establish results of fixed point for α_*-admissible mixed multi-valued mappings with respect to a function η. Also, we prove results of common fixed point for a pair (S, T) of multi-valued mappings that is α_*-admissible with respect to a function η in the setting of partial metric spaces.

Definition 3.6 ([62] Definition 2) *Let (X, p) be a partial metric space. $T : X \to X \cup CB^p(X)$ is called a mixed multi-valued mapping on X if T is a multi-valued mapping on X such that for each $x \in X$, $Tx \in X$ or $Tx \in CB^p(X)$.*

As said above, both a single-valued mapping $T : X \to X$ and a multi-valued mapping $T : X \to CB^p(X)$, are mixed multi-valued mappings. This approach is motivated, in part, by the fact that $CB^p(X)$ may be empty.

Now, we consider the family

$$\Psi = \{(\psi_1, \ldots, \psi_5) : \psi_i : [0, +\infty[\to [0, +\infty[, i = 1, \ldots, 5\}$$

such that

(i) ψ_2, ψ_5 are nondecreasing and ψ_4 is increasing;

(ii) $\psi_1(t), \psi_2(t), \psi_3(t) \leq \psi_4(t)$ for all $t > 0$;

(iii) $\psi_4(s + t) \leq \psi_4(s) + \psi_4(t)$ for all $s, t > 0$;

(iv) $\psi_1(t), \psi_2(t), \psi_5(t)$ are continuous in $t = 0$ and

$$\psi_1(0) = \psi_2(0) = \psi_5(0) = 0;$$

(v) $\sum_{n=1}^{+\infty} \psi_4^n(t) < +\infty$ for all $t > 0$.

The following lemma is obvious.

Lemma 3.4 *If $(\psi_1, \ldots, \psi_5) \in \Psi$, then $\psi_4(t) < t$ for all $t > 0$.*

Let (X, p) be a partial metric space and $\alpha, \eta : X \times X \to [0, +\infty[$ be two functions with η bounded. In the sequel we denote

$$\alpha_*(A, B) = \inf_{x \in A,\ y \in B} \alpha(x, y) \qquad \text{and} \qquad \eta_*(A, B) = \sup_{x \in A,\ y \in B} \eta(x, y)$$

for every $A, B \subset X$.

Definition 3.7 ([62] Definition 3) *Let (X, p) be a partial metric space, $T : X \to X \cup CB^p(X)$ a mixed multi-valued mapping and $\alpha : X \times X \to [0, +\infty[$ a function. We say that T is an α_*-admissible mixed multi-valued mapping if*

$$\alpha(x, y) \geq 1 \text{ implies } \alpha_*(Tx, Ty) \geq 1, \quad x, y \in X.$$

Definition 3.8 ([62] Definition 4) *Let (X, p) be a partial metric space, $S, T : X \to X \cup CB^p(X)$ be two mixed multi-valued mappings and $\alpha, \eta : X \times X \to [0, +\infty[$ be two functions with η bounded. We say that the pair (S, T) is α_*-admissible with respect to η if:*

$$\alpha(x, y) \geq \eta(x, y) \ \text{ implies } \ \alpha_*(Sx, Ty) \geq \eta_*(Sx, Ty), \quad x, y \in X.$$

We say that T is an α_*-admissible mixed multi-valued mapping with respect to η if the pair (T, T) is α_*-admissible with respect to η.

If we take, $\eta(x, y) = 1$ for all $x, y \in X$, then the definition of α_*-admissible mixed multi-valued mapping with respect to η reduces to Definition 3.7.

The following theorem is one of our main results.

Theorem 3.4 ([62] Theorem 2) *Let (X, p) be a 0-complete partial metric space and let $T : X \to X \cup CB^p(X)$ be a mixed multi-valued mapping. Assume that there exist $(\psi_1, \ldots, \psi_5) \in \Psi$ and two functions $\alpha, \eta : X \times X \to [0, +\infty[$ with η bounded, such that*

$$\inf_{u \in Tx} \eta(x, u) \leq \alpha(x, y) \quad \text{implies}$$

$$\begin{aligned} H(Tx, Ty) \ \leq \ \max\Big\{ & \psi_1(p(x, y)), \psi_2(p(x, Tx)), \psi_3(p(y, Ty)), \\ & \frac{\psi_4(p(x, Ty)) + \psi_5(p(y, Tx) - p(y, y))}{2} \Big\} \end{aligned} \tag{3.13}$$

for all $x, y \in X$.
Also suppose that the following assertions hold:

(i) T *is an* α_**-admissible mixed multi-valued mapping with respect to* η*;*

(ii) there exist $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$*;*

(iii) for a sequence $\{x_n\} \subset X$ *such that* $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x$ *as* $n \to +\infty$*, then either*

$$\inf_{u_n \in Ty_n} \eta(y_n, u_n) \leq \alpha(y_n, x) \quad \text{or} \quad \inf_{v_n \in Tz_n} \eta(z_n, v_n) \leq \alpha(z_n, x)$$

holds for all $n \in \mathbb{N}$ *where* $\{y_n\}$ *and* $\{z_n\}$ *are two given sequences such that* $y_n \in Tx_n$ *and* $z_n \in Ty_n$ *for all* $n \in \mathbb{N}$*.*

Then T *has a fixed point.*

Proof. By (ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that

$$\alpha(x_0, x_1) \geq \eta(x_0, x_1).$$

This implies that

$$\alpha(x_0, x_1) \geq \eta(x_0, x_1) \geq \inf_{y \in Tx_0} \eta(x_0, y).$$

If $x_0 = x_1$ or $x_1 \in Tx_1$, then x_1 is a fixed point of T. Assume that $x_1 \notin Tx_1$ and that Tx_1 is not a singleton. Therefore from (3.13), we have:

$$\begin{aligned}
0 < p(x_1, Tx_1) &\leq H(Tx_0, Tx_1) \\
&\leq \max\Big\{\psi_1(p(x_0, x_1)), \psi_2(p(x_0, Tx_0)), \psi_3(p(x_1, Tx_1)), \\
&\qquad \frac{\psi_4(p(x_0, Tx_1)) + \psi_5(p(x_1, Tx_0) - p(x_1, x_1))}{2}\Big\} \\
&\leq \max\Big\{\psi_1(p(x_0, x_1)), \psi_2(p(x_0, x_1)), \psi_3(p(x_1, Tx_1)), \\
&\qquad \frac{\psi_4(p(x_0, x_1)) + \psi_4(p(x_1, Tx_1))}{2}\Big\} \\
&\leq \max\{\psi_1(p(x_0, x_1)), \psi_2(p(x_0, x_1)), \psi_3(p(x_1, Tx_1)), \\
&\qquad \max\{\psi_4(p(x_0, x_1)), \psi_4(p(x_1, Tx_1))\}\} \\
&= \max\{\psi_4(p(x_0, x_1)), \psi_4(p(x_1, Tx_1))\}.
\end{aligned}$$

Now, if

$$\max\{\psi_4(p(x_0, x_1)), \psi_4(p(x_1, Tx_1))\} = \psi_4(p(x_1, Tx_1)),$$

then

$$0 < p(x_1, Tx_1) \leq H(Tx_0, Tx_1) \leq \psi_4(p(x_1, Tx_1)) < p(x_1, Tx_1)$$

which is a contradiction. Hence,

$$0 < p(x_1, Tx_1) \leq H(Tx_0, Tx_1) \leq \psi_4(p(x_0, x_1)).$$

If $q > 1$, then

$$0 < p(x_1, Tx_1) < qH(Tx_0, Tx_1) \leq q\psi_4(p(x_0, x_1)).$$

So there exists $x_2 \in Tx_1$ such that

$$0 < p(x_1, x_2) < qH(Tx_0, Tx_1) \leq q\psi_4(p(x_0, x_1)). \tag{3.14}$$

If $Tx_1 = \{x_2\}$ is a singleton, again by (3.13), we get

$$0 < p(x_1, x_2) \leq H(Tx_0, Tx_1) \leq \psi_4(p(x_0, x_1))$$

and so (3.14) holds.

Note that $x_1 \neq x_2$. Also, since T is α_*-admissible with respect to η, we have $\alpha_*(Tx_0, Tx_1) \geq \eta_*(Tx_0, Tx_1)$. This implies

$$\begin{aligned} \alpha(x_1, x_2) &\geq \alpha_*(Tx_0, Tx_1) \geq \eta_*(Tx_0, Tx_1) \\ &\geq \eta(x_1, x_2) \geq \inf_{y \in Tx_1} \eta(x_1, y). \end{aligned}$$

Therefore from (3.13), we have

$$\begin{aligned} H(Tx_1, Tx_2) &\leq \max\Big\{\psi_1(p(x_1, x_2)), \psi_2(p(x_1, Tx_1)), \psi_3(p(x_2, Tx_2)), \\ &\quad \frac{\psi_4(p(x_1, Tx_2)) + \psi_5(p(x_2, Tx_1) - p(x_2, x_2))}{2}\Big\} \\ &\leq \psi_4(p(x_1, x_2)). \end{aligned} \tag{3.15}$$

Put $t_0 = p(x_0, x_1) > 0$. Then from (3.14), we deduce that

$$p(x_1, x_2) < q\psi_4(t_0).$$

Now, since ψ_4 is increasing, we deduce $\psi_4(p(x_1, x_2)) < \psi_4(q\psi_4(t_0))$. Put

$$q_1 = \frac{\psi_4(q\psi_4(t_0))}{\psi_4(p(x_1, x_2))} > 1.$$

If $x_2 \in Tx_2$, then x_2 is a fixed point of T. Hence we suppose that $x_2 \notin Tx_2$. Then

$$0 < p(x_2, Tx_2) \leq H(Tx_1, Tx_2) < q_1 H(Tx_1, Tx_2).$$

So there exists $x_3 \in Tx_2$ (obviously $x_3 = Tx_2$ if Tx_2 is a singleton) such that

$$0 < p(x_2, x_3) < q_1 H(Tx_1, Tx_2)$$

and from (3.15), we get

$$\begin{aligned} 0 &< p(x_2, x_3) < q_1 H(Tx_1, Tx_2) \\ &\leq q_1 \psi_4(p(x_1, x_2)) = \psi_4(q\psi_4(t_0)). \end{aligned}$$

Again, since ψ_4 is increasing, then

$$\psi_4(p(x_2, x_3)) < \psi_4(\psi_4(q\psi_4(t_0))).$$

Put

$$q_2 = \frac{\psi_4(\psi_4(q\psi_4(t_0)))}{\psi_4(p(x_2, x_3))} > 1.$$

If $x_3 \in Tx_3$, then x_3 is a fixed point of T. Hence we assume that $x_3 \notin Tx_3$. Then

$$0 < p(x_3, Tx_3) \leq H(Tx_2, Tx_3) < q_2 H(Tx_2, Tx_3).$$

So there exists $x_4 \in Tx_3$ (obviously $x_4 = Tx_3$ if Tx_3 is a singleton) such that

$$0 < p(x_3, x_4) < q_2 H(Tx_2, Tx_3). \tag{3.16}$$

Clearly, $x_2 \neq x_3$. Again, since T is α_*-admissible with respect to η,

$$\begin{aligned} \alpha(x_2, x_3) &\geq \alpha_*(Tx_1, Tx_2) \geq \eta_*(Tx_1, Tx_2) \\ &\geq \eta(x_2, x_3) \geq \inf_{y \in Tx_2} \eta(x_2, y). \end{aligned}$$

Then from (3.13), we have

$$\begin{aligned} H(Tx_2, Tx_3) &\leq \max\Big\{\psi_1(p(x_2, x_3)), \psi_2(p(x_2, Tx_2)), \qquad (3.17) \\ &\quad \psi_3(p(x_3, Tx_3)), \frac{\psi_4(p(x_2, Tx_3)) + \psi_5(p(x_3, Tx_2) - p(x_3, x_3))}{2}\Big\} \\ &\leq \psi_4(p(x_2, x_3)). \end{aligned}$$

Thus from (3.16) and (3.17), we deduce that

$$\begin{aligned} 0 &< p(x_3, x_4) < q_2 H(Tx_2, Tx_3) \\ &\leq q_2 \psi_4(p(x_2, x_3)) = \psi_4(\psi_4(q\psi_4(t_0))). \end{aligned}$$

By continuing this process, we obtain a sequence $\{x_n\} \subset X$ such that

$$x_n \in Tx_{n-1},\ x_n \neq x_{n-1},\ \alpha(x_{n-1}, x_n) \geq \eta(x_{n-1}, x_n)$$

and

$$p(x_n, x_{n+1}) \leq \psi_4^{n-1}(q\psi_4(t_0))$$

for all $n \in \mathbb{N}$. Now for all $m > n$, we can write

$$p(x_n, x_m) \leq \sum_{k=n}^{m-1} p(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \psi_4^{k-1}(q\psi_4(t_0)).$$

Therefore $\{x_n\}$ is a 0-Cauchy sequence. Since, (X, p) is a 0-complete partial metric space, then there exists $z \in X$ such that

$$p(x_n, z) \to p(z, z) = 0 \text{ as } n \to +\infty.$$

Then from (iii), either

$$\inf_{u_n \in Ty_n} \eta(y_n, u_n) \leq \alpha(y_n, z) \quad \text{or} \quad \inf_{v_n \in Tz_n} \eta(z_n, v_n) \leq \alpha(z_n, z)$$

holds for all $n \in \mathbb{N}$ where $\{y_n\}$ and $\{z_n\}$ are two given sequences such that $y_n \in Tx_n$ and $z_n \in Ty_n$ for all $n \in \mathbb{N}$. Here, $x_{n-1} \in Tx_{n-2}$ and $x_n \in Tx_{n-1}$.

Therefore, either

$$\inf_{u_n \in Tx_{n-1}} \eta(x_{n-1}, u_n) \leq \alpha(x_{n-1}, z) \quad \text{or} \quad \inf_{v_n \in Tx_n} \eta(x_n, v_n) \leq \alpha(x_n, z)$$

holds for all $n \in \mathbb{N}$. If $p(z, Tz) > 0$, from (3.13), we have

$$\begin{aligned} p(z, Tz) &\leq H(Tx_{n-1}, Tz) + p(x_n, z) - p(x_n, x_n) \\ &\leq \max\Big\{\psi_1(p(x_{n-1}, z)), \psi_2(p(x_{n-1}, Tx_{n-1})), \psi_3(p(z, Tz)), \\ &\qquad \frac{\psi_4(p(x_{n-1}, Tz)) + \psi_5(p(z, Tx_{n-1}))}{2}\Big\} + p(x_n, z) \\ &\leq \max\Big\{\psi_1(p(x_{n-1}, z)), \psi_2(p(x_{n-1}, x_n)), \psi_3(p(z, Tz)), \\ &\qquad \frac{\psi_4(p(x_{n-1}, z) + p(z, Tz)) + \psi_5(p(z, x_n))}{2}\Big\} + p(x_n, z). \end{aligned}$$

or

$$\begin{aligned}
p(z,Tz) &\leq H(Tx_n,Tz)+p(x_{n+1},z)-p(x_{n+1},x_{n+1})\\
&\leq \max\Big\{\psi_1(p(x_n,z)),\psi_2(p(x_n,Tx_n)),\psi_3(p(z,Tz)),\\
&\quad \frac{\psi_4(p(x_n,Tz))+\psi_5(p(z,Tx_n))}{2}\Big\}+p(x_{n+1},z)\\
&\leq \max\Big\{\psi_1(p(x_n,z)),\psi_2(p(x_n,x_{n+1})),\psi_3(p(z,Tz)),\\
&\quad \frac{\psi_4(p(x_n,z)+p(z,Tz))+\psi_5(p(z,x_{n+1}))}{2}\Big\}+p(x_{n+1},z)
\end{aligned}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to +\infty$ in the above inequalities, we get

$$p(z,Tz) \leq \psi_4(p(z,Tz)) < p(z,Tz)$$

a contradiction. Thus $p(z,Tz)=0$. If Tz is a singleton, then $z=Tz$. If Tz is not a singleton, from $p(z,Tz)=0=p(z,z)$, by Remark 3.2, we deduce $z \in Tz$. Thus z is a fixed point of T. □

If in Theorem 3.4, we assume $\eta(x,y)=1$ for all $x,y \in X$, then we obtain the following corollary.

Corollary 3.2 *Let (X,p) be a 0-complete partial metric space and let $T: X \to X \cup CB^p(X)$ be a mixed multi-valued mapping. Assume that there exist $(\psi_1,\ldots,\psi_5) \in \Psi$ and a function $\alpha : X \times X \to [0,+\infty[$, such that*

$$\begin{aligned}
H(Tx,Ty) &\leq \max\Big\{\psi_1(p(x,y)),\psi_2(p(x,Tx)),\psi_3(p(y,Ty)),\\
&\quad \frac{\psi_4(p(x,Ty))+\psi_5(p(y,Tx)-p(y,y))}{2}\Big\}
\end{aligned}$$

for all $x,y \in X$ with $\alpha(x,y) \geq 1$. Also suppose the following assertions hold:

(i) T is an α_*-admissible mixed multi-valued mapping;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$;

(iii) for a sequence $\{x_n\} \subset X$ such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then either

$$\alpha(y_n,x) \geq 1 \quad \text{or} \quad \alpha(z_n,x) \geq 1$$

holds for all $n \in \mathbb{N}$ where $\{y_n\}$ and $\{z_n\}$ are two given sequences such that $y_n \in Tx_n$ and $z_n \in Ty_n$ for all $n \in \mathbb{N}$.

Then T has a fixed point.

Example 3.4 Let $X = \{1, 2, 3, 4\}$ and $p : X \times X \to [0, +\infty[$ be defined by:

- $p(1, 1) = p(2, 2) = p(4, 4) = 1/6$,
- $p(3, 3) = 0$,
- $p(1, 2) = p(1, 4) = p(2, 4) = p(3, 4) = 1/2$,
- $p(1, 3) = 1/4$,
- $p(2, 3) = 1/3$,
- $p(x, y) = p(y, x)$ for all $x, y \in X$.

Let $T : X \to CB^p(X)$ be defined by

$$T1 = \{3\},\ T2 = \{1\},\ T3 = \{3\} \quad \textit{and} \quad T4 = \{1, 4\}.$$

Clearly, (X, p) is a 0-complete partial metric space and Tx is a bounded closed subset of X for all $x \in X$.
Let $\alpha : X \times X \to [0, +\infty)$ be defined by: $\alpha(1, 1) = \alpha(1, 3) = \alpha(2, 3) = \alpha(3, 3) = \alpha(3, 1) = \alpha(3, 2) = 1$ and $\alpha(x, y) = 0$ otherwise.
Now, let $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5 : [0, +\infty[\to [0, +\infty[$ be defined by:

- $\psi_1(t) = t/2$,
- $\psi_2(t) = 2t/3$,
- $\psi_3(t) = t/2$,
- $\psi_4(t) = 3t/4$,
- $\psi_5(t) = 5t/6$,

then $(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5) \in \Psi$.

Now, we have:
$H(T1, T1) = H(\{3\}, \{3\}) = 0 \leq \psi_1(p(1, 1))$,
$H(T1, T3) = H(\{3\}, \{3\}) = 0 \leq \psi_1(p(1, 3))$,
$H(T2, T3) = H(\{1\}, \{3\}) = 1/4 \leq \psi_3(p(2, \{1\}))$,
$H(T3, T3) = H(\{3\}, \{3\}) = 0 \leq \psi_1(p(3, 3))$.

This implies

$$H(Tx,Ty) \leq \max \Big\{\psi_1(p(x,y)), \psi_2(p(x,Tx)), \psi_3(p(y,Ty)), \frac{\psi_4(p(x,Ty)) + \psi_5[p(y,Tx) - p(y,y)]}{2}\Big\}$$

for all $x, y \in X$ with $\alpha(x,y) \geq 1$. T is an α_*-admissible mixed multi-valued mapping and $x_0 = 1$ satisfies condition (ii). Now, we note that for a sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, we have $x = 3$ and this ensures that (iii) holds. Thus, by Corollary 3.2 the mixed multi-valued mapping T has a fixed point.

We note that

$$H(T2,T4) = \frac{1}{2} > \max \Big\{\psi_1(p(2,4)), \psi_2(p(2,T2)), \psi_3(p(4,T4)), \frac{\psi_4(p(2,T4)) + \psi_5(p(4,T2) - p(4,4))}{2}\Big\}.$$

3.4 Common Fixed Points of Multi-Valued Mappings

Let (X,p) be a partial metric space, let $\alpha, \eta : X \times X \to [0,+\infty[$ be two functions with η bounded and let $S, T : X \to 2^X$ be two multi-valued mappings on X. We denote

$$\Gamma(Sx,Ty) = \min\left\{\inf_{u\in Sx} \eta(x,u), \inf_{v\in Ty} \eta(y,v)\right\} = \Gamma(Ty,Sx).$$

Let $\Phi = \{(\psi_1,\ldots,\psi_5) : \psi_i : [0,+\infty[\to [0,+\infty[,\ i = 1,\ldots,5\}$ such that

(i) ψ_2, ψ_3 are nondecreasing and ψ_4, ψ_5 are increasing;

(ii) $\psi_1(t), \psi_2(t), \psi_3(t) \leq \min\{\psi_4(t), \psi_5(t)\}$ for all $t > 0$;

(iii) $\psi_i(s+t) \leq \psi_i(s) + \psi_i(t)$ $(i = 4, 5)$ for all $s, t > 0$;

(iv) $\psi_1(t), \psi_2(t)$ and $\psi_3(t)$ are continuous in $t = 0$ and $\psi_1(0) = \psi_2(0) = \psi_3(0) = 0$;

(v) $\sum_{n=1}^{+\infty} \psi_5^n(t) < +\infty$ for all $t > 0$;

(vi) $\psi_4(t) < t$ for all $t > 0$;

(vii) $\psi_4(\psi_5(t)) = \psi_5(\psi_4(t))$ for all $t > 0$.

The following theorem is our main result on the existence of common fixed point for multi-valued mappings.

Theorem 3.5 ([62] Theorem 3) *Let (X,p) be a 0-complete partial metric space and let $S,T : X \to X \cup CB^p(X)$ be two mixed multi-valued mappings on X. Assume that there exist $(\psi_1,\ldots,\psi_5) \in \Phi$ and two functions $\alpha,\eta : X \times X \to [0,+\infty[$ with η bounded, such that*

$$\begin{aligned} H(Sx,Ty) \quad \leq \quad & \max\Big\{\psi_1(p(x,y)), \psi_2(p(x,Sx)), \psi_3(p(y,Ty)), \\ & \frac{\psi_4(p(x,Ty)-p(x,x)) + \psi_5(p(y,Tx)-p(y,y))}{2}\Big\} \end{aligned} \tag{3.18}$$

for all $x,y \in X$ with $\alpha(x,y) \geq \Gamma(Sx,Ty)$. Also suppose the following assertions hold:

(i) the pair (S,T) is α_*-admissible with respect to η;

(ii) there exist $x_0 \in X$ and $x_1 \in Sx_0$ such that $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$;

(iii) $\alpha(x,x) \geq \Gamma(Sx,Tx)$ for all $x \in X$ which is a fixed point of S or T;

(iv) for a sequence $\{x_n\} \subset X$ such that $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then either

$$\inf_{u_n \in Sy_n} \eta(y_n,u_n) \leq \alpha(y_n,x) \quad or \quad \inf_{v_n \in Tz_n} \eta(z_n,v_n) \leq \alpha(z_n,x)$$

holds for all $n \in \mathbb{N}$ where $\{y_n\}$ and $\{z_n\}$ are two given sequences such that $y_n \in Tx_n$ and $z_n \in Sy_n$ for all $n \in \mathbb{N}$.

Then S and T have a common fixed point.

Proof. From (iii) and (3.18) it follows that the mixed multi-valued mappings S and T have the same fixed points. Let $x_0 \in X$ and $x_1 \in Sx_0$ be such that $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$, then

$$\alpha(x_0,x_1) \geq \eta(x_0,x_1) \geq \inf_{u \in Sx_0} \eta(x_0,u) \geq \Gamma(Sx_0,Tx_1).$$

If $x_0 = x_1$, then x_0 is a common fixed point of S and T. The same holds if $x_1 \in Tx_1$. Hence, we assume that $x_0 \neq x_1$ and $x_1 \notin Tx_1$. Assume that Tx_1 is not a singleton, from (3.18), we have

$$\begin{aligned}
0 < p(x_1, Tx_1) &\leq H(Sx_0, Tx_1) \\
&\leq \max\Big\{\psi_1(p(x_0, x_1)), \psi_2(p(x_0, Sx_0)), \psi_3(p(x_1, Tx_1)), \\
&\quad \frac{\psi_4(p(x_0, Tx_1) - p(x_0, x_0)) + \psi_5(p(x_1, Sx_0) - p(x_1, x_1))}{2}\Big\} \\
&\leq \max\Big\{\psi_1(p(x_0, x_1)), \psi_2(p(x_0, x_1)), \psi_3(p(x_1, Tx_1)), \\
&\quad \frac{\psi_4(p(x_0, x_1) + p(x_1, Tx_1) - p(x_1, x_1) - p(x_0, x_0))}{2}\Big\} \\
&\leq \max\{\psi_1(p(x_0, x_1)), \psi_2(p(x_0, x_1)), \psi_3(p(x_1, Tx_1)), \\
&\quad \max\{\psi_4(p(x_0, x_1)), \psi_4(p(x_1, Tx_1))\}\} \\
&= \max\{\psi_4(p(x_0, x_1)), \psi_4(p(x_1, Tx_1))\}.
\end{aligned}$$

Now, if

$$\max\{\psi_4(p(x_0, x_1)), \psi_4(p(x_1, Tx_1))\} = \psi_4(p(x_1, Tx_1)),$$

then

$$\begin{aligned}
0 &< p(x_1, Tx_1) \leq H(Sx_0, Tx_1) \\
&\leq \psi_4(p(x_1, Tx_1)) < p(x_1, Tx_1)
\end{aligned}$$

which is a contradiction. Hence,

$$\max\{\psi_4(p(x_0, x_1)), \psi_4(p(x_1, Tx_1))\} = \psi_4(p(x_0, x_1)).$$

If $q > 1$, then

$$0 < p(x_1, Tx_1) \leq H(Sx_0, Tx_1) < qH(Sx_0, Tx_1)$$

and hence there exists $x_2 \in Tx_1$ such that

$$0 < p(x_1, x_2) < qH(Sx_0, Tx_1) \leq q\psi_4(p(x_0, x_1)). \tag{3.19}$$

If $Tx_1 = \{x_2\}$ is a singleton, again by (3.18), we get

$$0 < p(x_1, x_2) \leq H(Sx_0, Tx_1) \leq \psi_4(p(x_0, x_1))$$

and so (3.19) holds. Note that $x_1 \neq x_2$. Also, since the pair (S, T) is α_*-admissible with respect to η, then $\alpha_*(Sx_0, Ty_1) \geq \eta_*(Sx_0, Ty_1)$. This implies

$$\begin{aligned}
\alpha(x_1, x_2) &\geq \alpha_*(Sx_0, Tx_1) \geq \eta_*(Sx_0, Tx_1) \\
&\geq \eta(x_1, x_2) \geq \inf_{y \in Tx_1} \eta(x_1, y) \\
&\geq \Gamma(Sx_2, Tx_1).
\end{aligned}$$

If $x_2 \in Sx_2$, then x_2 is a common fixed point of S and T. Assume that $x_2 \notin Sx_2$ and that Sx_2 is not a singleton, from (3.18), we have

$$\begin{aligned}
0 &< p(x_2, Sx_2) \leq H(Sx_2, Tx_1) \\
&\leq \max\Big\{\psi_1(p(x_2, x_1)), \psi_2(p(x_2, Sx_2)), \psi_3(p(x_1, Tx_1)), \\
&\quad \frac{\psi_4(p(x_2, Tx_1) - p(x_2, x_2)) + \psi_5(p(x_1, Sx_2) - p(x_1, x_1))}{2}\Big\} \\
&\leq \max\Big\{\psi_1(p(x_1, x_2)), \psi_2(p(x_2, Sx_2)), \psi_3(p(x_1, x_2)), \\
&\quad \frac{\psi_5(p(x_1, x_2) + p(x_2, Sx_2) - p(x_2, x_2) - p(x_1, x_1))}{2}\Big\} \\
&\leq \max\{\psi_5(p(x_1, x_2)), \psi_5(p(x_2, Sx_2))\}.
\end{aligned}$$

Now, if

$$\max\{\psi_5(p(x_1, x_2)), \psi_5(p(x_2, Sx_2))\} = \psi_5(p(x_2, Sx_2)),$$

then

$$\begin{aligned}
0 &< p(x_2, Sx_2) \leq H(Sx_2, Tx_1) \\
&\leq \psi_5(p(x_2, Sx_2)) < p(x_2, Sx_2)
\end{aligned}$$

which is a contradiction. Hence,

$$0 < p(x_2, Sx_2) \leq H(Sx_2, Tx_1) \leq \psi_5(p(x_1, x_2)). \tag{3.20}$$

The same is worth also if Sx_2 is a singleton. Put $t_0 = p(x_0, x_1)$. Then from (3.19), we have $p(x_1, x_2) < q\psi_4(t_0)$ where $t_0 > 0$. Now, since ψ_5 is increasing, then $\psi_5(p(x_1, x_2)) < \psi_5(q\psi_4(t_0))$. Put

$$q_1 = \frac{\psi_5(q\psi_4(t_0))}{\psi_5(p(x_1, x_2))} > 1.$$

Since $x_2 \in Tx_1$ or $x_2 = Tx_1$, we have

$$0 < p(x_2, Sx_2) \leq H(Sx_2, Tx_1) < q_1 H(Sx_2, Tx_1)$$

and hence there exists $x_3 \in Sx_2$ or $x_3 = Sx_2$ such that

$$0 < p(x_2, x_3) \leq q_1 H(Sx_2, Tx_1).$$

Now, from (3.20), we deduce

$$\begin{aligned}
0 &< p(x_2, x_3) < q_1 H(Sx_2, Tx_1) \\
&\leq q_1 \psi_5(p(x_1, x_2)) = \psi_5(q\psi_4(t_0)).
\end{aligned}$$

Clearly, $x_2 \neq x_3$. Again, since the pair (S, T) is α_*-admissible with respect to η, then

$$\begin{aligned}\alpha(x_2, x_3) &\geq \alpha_*(Tx_1, Sx_2) \geq \eta_*(Tx_1, Sx_2) \\ &\geq \eta(x_2, x_3) \geq \inf_{y \in Sx_2} \eta(x_2, y) \geq \Gamma(Sx_2, Tx_3).\end{aligned}$$

If $x_3 \in Tx_3$ or $x_3 = Tx_3$, then x_3 is a common fixed point of S and T. Assume that $x_3 \notin Tx_3$. Now, from (3.18) we deduce

$$\begin{aligned}0 &< p(x_3, Tx_3) \leq H(Sx_2, Tx_3) \\ &\leq \max\Big\{\psi_1(p(x_2, x_3)), \psi_2(p(x_2, Sx_2)), \psi_3(p(x_3, Tx_3)), \\ &\quad \frac{\psi_4(p(x_2, Tx_3) - p(x_2, x_2)) + \psi_5(p(x_3, Sx_2) - p(x_3, x_3))}{2}\Big\} \\ &\leq \max\Big\{\psi_1(p(x_2, x_3)), \psi_2(p(x_2, x_3)), \psi_3(p(x_3, Tx_3)), \\ &\quad \frac{\psi_4(p(x_2, x_3) + p(x_3, Tx_3) - p(x_3, x_3) - p(x_2, x_2))}{2}\Big\} \\ &\leq \max\{\psi_4(p(x_2, x_3)), \psi_4(p(x_3, Tx_3))\}.\end{aligned}$$

If $\max\{\psi_4(p(x_2, x_3)), \psi_4(p(x_3, Tx_3))\} = \psi_4(p(x_3, Tx_3))$, then

$$0 < p(x_3, Tx_3) \leq H(Sx_2, Tx_3) \leq \psi_4(p(x_3, Tx_3)) < p(x_3, Tx_3)$$

which is a contradiction. Hence,

$$\max\{\psi_4(p(x_2, x_3)), \psi_4(p(x_3, Tx_3))\} = \psi_4(p(x_2, x_3))$$

and so

$$0 < p(x_3, Tx_3) \leq H(Sx_2, Tx_3) \leq \psi_4(p(x_2, x_3)). \tag{3.21}$$

Again, since ψ_4 is increasing, we deduce that

$$\psi_4(p(x_2, x_3)) < \psi_4(\psi_5(q\psi_4(t_0))).$$

Put

$$q_2 = \frac{\psi_4(\psi_5(q\psi_4(t_0)))}{\psi_4(p(x_2, x_3))} > 1.$$

Then

$$0 < p(x_3, Tx_3) \leq H(Sx_2, Tx_3) < q_2 H(Sx_2, Tx_3)$$

and hence there exists $x_4 \in Tx_3$ or $x_4 = Tx_3$ such that

$$0 < p(x_3, x_4) < q_2 H(Sx_2, Tx_3) \leq q_2\psi_4(p(x_2, x_3)). \tag{3.22}$$

Now, from (3.21) and (3.22), we deduce that

$$\begin{aligned} 0 &< p(x_3, x_4) < q_2 H(Sx_2, Tx_3) \\ &\leq q_2\psi_4(p(x_2, x_3)) = \psi_4(\psi_5(q\psi_4(t_0))). \end{aligned}$$

By continuing this process, we obtain a sequence $\{x_n\}$ in X such that $x_{2n} \in Tx_{2n-1}$, $x_{2n+1} \in Sx_{2n}$ and

$$p(x_{2n-1}, x_{2n}) \leq (\psi_4\psi_5)^{n-1}(q\psi_4(t_0)) \text{ and}$$

$$p(x_{2n}, x_{2n+1}) \leq \psi_5[(\psi_4\psi_5)^{n-1}(q\psi_4(t_0))].$$

Now, for all $m > n$, we can write

$$\begin{aligned} p(x_{2n}, x_{2m}) &\leq \sum_{k=n}^{m-1} p(x_{2k}, x_{2k+1}) + \sum_{k=n}^{m-1} p(x_{2k+1}, x_{2k+2}) \\ &\leq \sum_{k=n}^{m-1} \psi_5^k(\psi_4^{k-1}(q\psi_4(t_0))) + \sum_{k=n}^{m-1} \psi_5^k(\psi_4^k(q\psi_4(t_0))) \\ &\leq 2\sum_{k=n}^{m-1} \psi_5^k(q\psi_4(t_0)). \end{aligned}$$

Since

$$\sum_{k=1}^{+\infty} \psi_5^k(q\psi_4(t_0)) < +\infty,$$

we get

$$\lim_{n\to+\infty} p(x_{2n}, x_{2m}) = 0.$$

Similary, we obtain

$$\lim_{n\to+\infty} p(x_{2n+1}, x_{2m+1}) = 0, \ \lim_{n\to+\infty} p(x_{2n+1}, x_{2m}) = 0,$$

$$\lim_{n\to+\infty} p(x_{2n}, x_{2m+1}) = 0.$$

This implies that $\lim_{n,m\to+\infty} p(x_n, x_m) = 0$ and so $\{x_n\}$ is a 0-Cauchy sequence. Since (X, p) is a 0-complete partial metric space, then there exists $z \in X$ with $p(z, z) = 0$ such that $x_n \to z$ as $n \to +\infty$. Then from (ii) either

$$\inf_{u\in Sy_n} \eta(y_n, u) \leq \alpha(y_n, z) \ \text{ or } \ \inf_{v\in Tz_n} \eta(z_n, v) \leq \alpha(z_n, z)$$

holds for all $n \in \mathbb{N}$ where $\{y_n\}$ and $\{z_n\}$ are two given sequences such that $y_n \in Tx_n$ and $z_n \in Sy_n$ for all $n \in \mathbb{N}$. Here, $x_{2n} \in Tx_{2n-1}$ and $x_{2n+1} \in Sx_{2n}$. Therefore, either

$$\inf_{u \in Sx_{2n}} \eta(x_{2n}, u) \leq \alpha(x_{2n}, z) \quad \text{or} \quad \inf_{v \in Tx_{2n+1}} \eta(x_{2n+1}, v) \leq \alpha(x_{2n+1}, z)$$

holds for all $n \in \mathbb{N}$. So from (3.18) and $p(z, z) = 0$, we have

$$\begin{aligned}
0 < p(z, Tz) \quad &\leq \quad H(Sx_{2n}, Tz) + p(x_{2n+1}, z) - p(x_{2n+1}, x_{2n+1}) \\
&\leq \quad \max\Big\{\psi_1(p(x_{2n}, z)), \psi_2(p(x_{2n}, Sx_{2n})), \psi_3(p(z, Tz)), \\
&\qquad \frac{\psi_4(p(x_{2n}, Tz) - p(x_{2n}, x_{2n})) + \psi_5(p(z, Sx_{2n}))}{2}\Big\} \\
&\qquad + p(x_{2n+1}, z)
\end{aligned}$$

or

$$\begin{aligned}
0 < p(z, Sz) \quad &\leq \quad H(Tx_{2n+1}, Sz) + p(x_{2n+2}, z) - p(x_{2n+2}, x_{2n+2}) \\
&\leq \quad \max\Big\{\psi_1(p(x_{2n+1}, z)), \psi_2(p(z, Sz)), \psi_3(p(x_{2n+1}, Tx_{2n+1})), \\
&\qquad \frac{\psi_4(p(z, Tx_{2n+1})) + \psi_5(p(x_{2n+1}, Sz) - p(x_{2n+1}, x_{2n+1}))}{2}\Big\} \\
&\qquad + p(x_{2n+2}, z)
\end{aligned}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to +\infty$ in above inequalities we get

$$p(z, Tz) \leq \psi_4(p(z, Tz)) \quad \text{or} \quad p(z, Sz) \leq \psi_5(p(z, Sz))$$

and hence $p(z, Tz) = 0$ or $p(z, Sz) = 0$. This implies that z is a fixed point of T or S and hence z is a common fixed point of the mixed multi-valued mappings S and T. $\square$

3.5 Fixed Points of Partial-Special Multi-Valued Mappings

Now we introduce a notion called partial-special multi-valued mapping. For this type of partial-special multi-valued mappings we have obtained a fixed point theorem that generalizes a Geraghty's fixed point theorem for multi-valued mappings.

Definition 3.9 *Let (X,p) be a partial metric space, a multi-valued mapping $T : X \to CB^p(X)$ is called partial-special multi-valued mapping if*

$$\inf_{y\in Tx}\{p(x,y)+p(y,z)\} = p(x,Tx)+p(z,Tx), \quad \forall\ x,y\in X. \tag{3.23}$$

It is clear that every single valued mapping, in a partial metric space, is partial-special multi-valued mapping, also there exist some mappings that are partial-special multi-valued but not single-valued.

Example 7. Let $X = \left\{\frac{1}{3},\frac{1}{9},\cdots,\frac{1}{3^n},\cdots\right\}\cup\{0,1\}$, $d(x,y) = \begin{cases} 1 & x\neq y \\ 0 & x=y. \end{cases}$
Define $p(x,y) = d(x,y)+c$ with $c\geq 0$ arbitrary.
Define mapping $Tx : X \to CB^p(X)$,

$$Tx = \begin{cases} \left\{\frac{1}{3^{n+1}}\right\} & \text{if } x=\frac{1}{3^n}, \quad n=1,2,\cdots \\ \{0\} & \text{if } x=0 \\ \left\{0,\frac{1}{3}\right\} & \text{if } x=1. \end{cases}$$

The mapping T is partial-metric multi-valued mapping, it is possible to check (3.23) for every couple $x,y\in X$. It is clear that above example is partial-special multi-valued mapping but not single-valued. Now we prove another result in this work.

Theorem 3.6 *Let (X,p) be a complete partial metric space and let $T : X \to CB^p(X)$ be partial-special multi-valued mapping such that*

$$\begin{aligned} H_p(Tx,Ty) \leq \alpha(p(x,y))p(x,y) &+ \beta(p(x,y))[p(x,Tx)+p(y,Ty)] \\ &+ \gamma(p(x,y))[p(x,Ty)+p(y,Tx)] \end{aligned}$$

for all $x,y\in X$, where α,β,γ are mappings from $[0,+\infty[$ into $[0,1[$ such that

$$\frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)}\in S \quad \text{and } \beta(t)\geq\gamma(t) \text{ for all } t\in[0,+\infty[.$$

Then T has a fixed point.

Proof. Define a function α' from $[0,+\infty[$ into $[0,1[$ by

$$\alpha'(t) = \frac{\alpha(t)+1-2\beta(t)-2\gamma(t)}{2} \quad \text{for all } t\in[0,+\infty[.$$

Then we have

1. $\alpha(t) < \alpha'(t)$ for all $t \in [0, +\infty[$;
2. $\frac{\alpha'+\beta+\gamma}{1-(\beta+\gamma)} \in S$;
3. for $x, y \in X$ and $u \in Tx$, there exists $v \in Ty$ such that

$$\begin{aligned} p(u,v) \leq \alpha'(p(x,y))p(x,y) \quad &+ \quad \beta(p(x,y))[p(x,Tx) + p(y,Ty)] \\ &+ \quad \gamma(p(x,y))[p(x,Ty) + p(y,Tx)]. \end{aligned}$$

Putting $u = y$ in 3., we obtain that:

4. For $x \in X$ and $y \in Tx$ there exists $v \in Ty$ such that

$$\begin{aligned} p(v,y) \leq \alpha'(p(x,y))p(x,y) \quad &+ \quad \beta(p(x,y))[p(x,Tx) + p(y,Ty)] \\ &+ \quad \gamma(p(x,y))[p(x,Ty) + p(y,Tx)]. \end{aligned}$$

Hence, we can define a sequence $\{x_n\}_{n\in\mathbb{N}}$ which satisfies

$$x_{n+1} \in Tx_n, \; x_{n+1} \neq x_n$$

and

$$\begin{aligned} p(x_{n+2}, x_{n+1}) \quad &\leq \quad \alpha'(p(x_{n+1}, x_n))p(x_{n+1}, x_n) \\ &+ \quad \beta(p(x_{n+1}, x_n))[p(x_n, Tx_n) + p(x_{n+1}, Tx_{n+1})] \\ &+ \quad \gamma(p(x_{n+1}, x_n))[p(x_n, Tx_{n+1}) + p(x_{n+1}, Tx_n)] \end{aligned}$$

for all $n \in \mathbb{N}$. Observing that

$$\begin{aligned} p(x_n, Tx_{n+1}) + p(x_{n+1}, Tx_n) \quad &\leq \quad p(x_n, x_{n+2}) + p(x_{n+1}, x_{n+1}) \\ &\leq \quad p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) \end{aligned}$$

it follows that

$$\begin{aligned} &p(x_{n+2}, x_{n+1}) \\ &\leq \frac{\alpha'(p(x_{n+1}, x_n)) + \beta(p(x_{n+1}, x_n)) + \gamma(p(x_{n+1}, x_n))}{1 - (\beta(p(x_{n+1}, x_n)) + \gamma(p(x_{n+1}, x_n)))} p(x_{n+1}, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$. We show that $\{x_n\}$ is a Cauchy sequence. To this end, we break the argument into two Steps.

Step 1: $\lim_{n\to+\infty} p(x_n, x_{n+1}) = 0$.

Since

$$\frac{\alpha'(t)+\beta(t)+\gamma(t)}{1-(\beta(t)+\gamma(t))} < 1 \quad \textit{for all } t,$$

$\{p(x_n, x_{n+1})\}$ is decreasing and bounded below, so

$$\lim_{n\to+\infty} p(x_n, x_{n+1}) = r \geq 0.$$

Assume $r > 0$. Then we have

$$\frac{p(x_{n+1}, x_{n+2})}{p(x_n, x_{n+1})} \leq \frac{\alpha'(p(x_n, x_{n+1})) + \beta(p(x_n, x_{n+1})) + \gamma(p(x_n, x_{n+1}))}{1-(\beta(p(x_n, x_{n+1})) + \gamma(p(x_n, x_{n+1})))} < 1,$$

$n = 1, 2, ...$

By letting $n \to +\infty$, we see that

$$1 \leq \lim_{n\to+\infty} \frac{\alpha'(p(x_n, x_{n+1})) + \beta(p(x_n, x_{n+1})) + \gamma(p(x_n, x_{n+1}))}{1-(\beta(p(x_n, x_{n+1})) + \gamma(p(x_n, x_{n+1})))} \leq 1.$$

On the other hand, we have $\frac{\alpha'+\beta+\gamma}{1-(\beta+\gamma)} \in S$. Therefore $r = 0$. This is a contradiction, hence, we prove Step 1.

Step 2: $\{x_n\}$ is a 0-Cauchy sequence.
Assume

$$\limsup_{n,m\to+\infty} p(x_n, x_m) > 0.$$

By triangle inequality for positive integer numbers n, m and for $y \in Tx_m$, we obtain

$$p(x_n, x_m) \leq p(x_n, y) + p(y, x_m) - p(y, y).$$

This means that for every positive integer numbers m, n, with using of relation (3.23), we have

$$\begin{aligned}
p(x_n, x_m) &\leq \inf\{p(x_n, y) + p(y, x_m) - p(y, y)\}\\
&\leq \inf\{p(x_n, y) + p(y, x_m)\} = p(x_m, Tx_m) + p(x_n, Tx_m)\\
&\leq p(x_m, x_{m+1}) + p(x_n, x_{n+1}) + p(x_{n+1}, Tx_m)\\
&\leq H_p(Tx_m, Tx_n) + p(x_n, x_{n+1}) + p(x_m, x_{m+1})\\
&\leq \alpha(p(x_n, x_m))p(x_n, x_m)\\
&\quad +\beta(p(x_n, x_m))[p(x_n, Tx_n) + p(x_m, Tx_m)]\\
&\quad +\gamma(p(x_n, x_m))[p(x_n, Tx_m) + p(x_m, Tx_n)]\\
&\quad +p(x_n, x_{n+1}) + p(x_m, x_{m+1})\\
&= \alpha(p(x_n, x_m))p(x_n, x_m)\\
&\quad +\beta(p(x_n, x_m))[p(x_n, x_{n+1}) + p(x_m, x_{m+1})]\\
&\quad +\gamma(p(x_n, x_m))[p(x_n, x_{m+1}) + p(x_m, x_{n+1})]\\
&\quad +p(x_n, x_{n+1}) + p(x_m, x_{m+1})\\
&\leq \alpha(p(x_n, x_m))p(x_n, x_m)\\
&\quad +\beta(p(x_n, x_m))[p(x_n, x_{n+1}) + p(x_m, x_{m+1})]\\
&\quad +\gamma(p(x_n, x_m))[p(x_n, x_m) + p(x_m, x_{m+1}) - p(x_m, x_m)]\\
&\quad +\gamma(p(x_n, x_m))[p(x_m, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n)]\\
&\quad +p(x_n, x_{n+1}) + p(x_m, x_{m+1})\\
&\leq \alpha(p(x_n, x_m))p(x_n, x_m)\\
&\quad +\beta(p(x_n, x_m))[p(x_n, x_{n+1}) + p(x_m, x_{m+1})]\\
&\quad +\gamma(p(x_n, x_m))[2p(x_n, x_m) + p(x_n, x_{n+1}) + p(x_m, x_{m+1})] +\\
&\quad +p(x_n, x_{n+1}) + p(x_m, x_{m+1}).
\end{aligned}$$

Then

$$\begin{aligned}
&p(x_n, x_m) - \alpha(p(x_n, x_m))p(x_n, x_m) - 2\gamma(p(x_n, x_m))p(x_n, x_m)\\
\leq\ &\beta(p(x_n, x_m))[p(x_n, x_{n+1}) + p(x_m, x_{m+1})]\\
&+\gamma(p(x_n, x_m))[p(x_n, x_{n+1}) + p(x_m, x_{m+1})]\\
&+p(x_n, x_{n+1}) + p(x_m, x_{m+1})
\end{aligned}$$

and hence:

$$p(x_n, x_m) \leq \frac{[\beta(p(x_n, x_m)) + \gamma(p(x_n, x_m))][p(x_n, x_{n+1}) + p(x_m, x_{m+1})] + p(x_n, x_{n+1}) + p(x_m, x_{m+1})}{1 - [\alpha(p(x_n, x_m)) + 2\gamma(p(x_n, x_m))]}.$$

Under the assumption

$$\limsup_{n,m\to+\infty} p(x_n, x_m) > 0,$$

it follows by Step 1, that

$$\limsup_{n,m\to+\infty} \frac{1}{1-[\alpha(p(x_n,x_m)) + 2\gamma(p(x_n,x_m))]} = +\infty$$

for which

$$\limsup_{n,m\to+\infty} [\alpha(p(x_n,x_m)) + 2\gamma(p(x_n,x_m))] = 1 \tag{3.24}$$

On the other hand, since

$$\frac{\alpha(t)+\beta(t)+\gamma(t)}{1-(\beta(t)+\gamma(t))} < 1 \tag{3.25}$$

then $\beta(t) + \gamma(t) < \frac{1}{2}$, for all $t \in [0, +\infty)$.
Hence, since $\beta(t) \geq \gamma(t)$, for all $t \in [0, +\infty)$, by using (3.24) and (3.25)

$$\begin{aligned}
&\limsup_{n,m\to+\infty} \frac{\alpha(p(x_n,x_m)) + \beta(p(x_n,x_m)) + \gamma(p(x_n,x_m))}{1-[\beta(p(x_n,x_m)) + \gamma(p(x_n,x_m))]} \\
&\quad\geq \limsup_{n,m\to+\infty} \frac{\alpha(p(x_n,x_m)) + 2\gamma(p(x_n,x_m))}{1-[\beta(p(x_n,x_m)) + \gamma(p(x_n,x_m))]} \\
&\quad\geq \limsup_{n,m\to+\infty} [\alpha(p(x_n,x_m)) + 2\gamma(p(x_n,x_m))] = 1.
\end{aligned} \tag{3.26}$$

Now since

$$\frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)} \in S,$$

then using (3.26), we have

$$\limsup_{n,m\to+\infty} \frac{\alpha(p(x_n,x_m)) + \beta(p(x_n,x_m)) + \gamma(p(x_n,x_m))}{1-[\beta(p(x_n,x_m)) + \gamma(p(x_n,x_m))]} = 1.$$

It follows that

$$\limsup_{n,m\to+\infty} p(x_n, x_m) = 0$$

which is a contradiction. Thus Step 2 is proved.

By completeness of X, there exists $x^* \in X$ such that

$$\lim_{n\to+\infty} p(x_n, x^*) = p(x^*, x^*) = 0.$$

Now, we have

$$\begin{aligned}
p(x^*, Tx^*) &\leq p(x^*, x_{n+1}) + (x_{n+1}, Tx^*) \\
&\leq p(x^*, x_{n+1}) + H_p(Tx_n, Tx^*) \\
&\leq p(x^*, x_{n+1}) + \alpha(p(x_n, x^*))p(x_n, x^*) \\
&\quad +\beta(p(x_n, x^*))[p(x_n, Tx_n) + p(x^*, Tx^*)] \\
&\quad +\gamma(p(x_n, x^*))[p(x_n, Tx^*) + p(x^*, Tx_n)]
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned}
p(x^*, Tx^*) &\leq p(x^*, x_{n+1}) + \alpha(p(x_n, x^*))p(x_n, x^*) \\
&\quad +[\beta(p(x_n, x^*)) + \gamma(p(x_n, x^*))][p(x_n, x_{n+1}) \\
&\quad +p(x^*, Tx^*) + p(x_n, Tx^*) + p(x^*, x_{n+1})].
\end{aligned}$$

On the other hand, since $\beta(t) + \gamma(t) < \frac{1}{2}$, for all $t \in [0, +\infty)$, then we have

$$\begin{aligned}
p(x^*, Tx^*) &< p(x^*, x_{n+1}) + \alpha(p(x_n, x^*))p(x_n, x^*) \\
&\quad +\frac{1}{2}[p(x_n, x_{n+1}) + p(x^*, Tx^*) + p(x_n, Tx^*) + p(x^*, x_{n+1})].
\end{aligned}$$

For $n \to +\infty$ it follows $p(x^*, Tx^*) < p(x^*, Tx^*)$, absurd. Then

$$p(x^*, Tx^*) = 0 = p(x^*, x^*).$$

We know that Tx^* is closed then, by Remark 3.2, we get $x^* \in Tx^*$. $\square$

Remark 3.5 *We observe that in previous theorem, we suppose $x \neq y$. Indeed in partial metric space it can be $p(x,x) \neq 0$. So, if $x = y$, it isn't possible to build the sequence that satisfies the contraction condiction. If we define* $p(x,y) = \begin{cases} d(x,y) & \text{if } x \neq y \\ 0 & \text{if } x = y, \end{cases}$
then the contraction condiction is always satisfied and we obtain the following note result in ordinary metric space:

Theorem 3.7 *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a special multi-valued mapping such that*

$$\begin{aligned} H_d(Tx, Ty) \leq \alpha(d(x,y))d(x,y) &+ \beta(d(x,y))[D(x,Tx) + D(y,Ty)] \\ &+ \gamma(d(x,y))[D(x,Ty) + D(y,Tx)] \end{aligned}$$

for all $x, y \in X$, where $D(x, A) = \inf\{d(x, a) : a \in A\}$ and where α, β, γ are mappings from $[0, +\infty[$ into $[0, 1[$ such that

$$\frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} \in S$$

and $\beta(t) \geq \gamma(t)$ for all $t \in [0, +\infty)$. Then T has a fixed point.

Corollary 3.3 *Let $T : X \to X$ be a mapping such that*

$$\begin{aligned} p(Tx, Ty) \leq \alpha(p(x,y))p(x,y) &+ \beta(p(x,y))[p(x,Tx) + p(y,Ty)] \\ &+\gamma(p(x,y))[p(x,Ty) + p(y,Tx)] \end{aligned}$$

for all $x, y \in X$, where α, β, γ, are mappings from $[0, +\infty[$ into $[0, 1[$ such that

$$\frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} \in S$$

and $\beta(t) \geq \gamma(t)$ for all $t \in [0, +\infty[$. Then T has a fixed point.

We observed that every single-valued mapping, in a partial metric space, is a partial-special multi-valued mapping. Then, putting $\beta = \gamma = 0$ in Theorem 2, we obtain the following corollary that is a partial-special multi-valued version of Geraghty's fixed point theorem.

Corollary 3.4 *Let (X, p) be a complete partial metric space and let $T : X \to CB^p(X)$ be a partial-special multi-valued mapping, $\alpha \in S$ and let*

$$H_p(Tx, Ty) \leq \alpha(p(x, y))p(x, y)$$

for all $x, y \in X$. Then T has a fixed point.

Corollary 3.5 *Let (X, p) be a complete partial metric space and let $T : X \to CB^p(X)$ be a partial-special multi-valued mapping such that*

$$H_p(Tx, Ty) \leq \beta(p(x, y))[D(x, Tx) + D(y, Ty)]$$

for all $x, y \in X$, where β is a mapping from $[0, +\infty[$ into $[0, \frac{1}{2}[$ such that $\frac{\beta}{1-\beta} \in S$. Then T has a fixed point.

Corollary 3.6 *Let (X, p) be a complete partial metric space and let $T : X \to CB^p(X)$ be a partial-special multi-valued mapping such that*

$$H_p(Tx, Ty) \leq \alpha(p(x, y))p(x, y) + \beta(p(x, y))[D(x, Tx) + D(y, Ty)]$$

for all $x, y \in X$, where α, β are mappings from $[0, +\infty[$ into $[0, 1[$ such that $\frac{\alpha+\beta}{1-\beta} \in S$. Then T has a fixed point.

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